R. B. HAYWARD

ROBERT BALDWIN HAYWARD (1829–1903), second President of our Association, was educated at University College, London, and St. John's College, Cambridge. In the Tripos of 1850 he was 4th Wrangler, and in 1852 was elected a Fellow of St. John's along with Joseph Wolstenholme, whose *Problems* are perhaps not yet forgotten. For a short time Hayward held a post in the University of Durham, but in 1859 he became a master at Harrow, where he remained till 1893. What he did for the teaching of mathematics at Harrow can be read in Mr. Siddons' Presidential Address, " Progress ", (*Gazette*, XX, No. 238, May 1936).

In 1876 Hayward was elected into the Fellowship of the Royal Society, a distinction which rarely falls to the lot of the schoolmaster. Of his published papers, that which seems most likely to perpetuate his name gives the familiar formula for the rate of change of a vector, such as angular momentum, when the frame of reference is in motion (1856).

Hayward's name appears in our first List of Members (1871) and is to be found prominently in the early annual Reports. He served on the committees which drew up our reforming syllabuses, particularly the syllabus of elementary geometry, and the early records of the Association suggest that his influence was second only to Levett's in the long fight for the reform of geometrical teaching.

In 1878, the first President of the Association, Hirst, who had held office from 1871, resigned, and was succeeded by Hayward. For eleven years he guided the affairs of the Association, and it would appear to have been his influence which caused the Association to widen its scope to advocate reform in the teaching of all branches of elementary mathematics ; certainly he was the prime mover in the preparation of the syllabus of linear dynamics. On his retirement from the President's chair in 1889, his fellow members deplored his decision to retire in terms which leave no doubt about their belief that Hayward had been a great driving force and an indefatigable worker in the severe struggle against the opponents of reform.

F

A VERY SHORT HISTORY OF MATHEMATICS.*

By W. W. O. Slessenger and A. R. Curtis.

This paper was read to the Adams Society (St. John's College, Cambridge, Mathematical Society) at their twenty-fifth anniversary dinner, Michaelmas Term, 1948.

Mathematics is very much older than History, which begins † in + 1066, as is well known ; for the first mathematician of any note was a Greek named Zeno, who was born in − 494, just 1560 years‡ earlier. Zeno is memorable for proving three theorems : (i) that motion is impossible ; (ii) that Achilles can never catch the tortoise (he failed to notice that this follows from his first theorem) ; and (iii) that half the time may be equal to double the time. This was not considered a very good start by the other Greeks, so they turned their attention to Geometry.

Euclid, about − 300, invented Geometry, including Pythagoras' theorem, which is how it got the name. He also invented parallel lines, which have really been of more use to railways than to mathematicians. Most people already know more about Euclid than we do.

Archimedes (− 286 to − 211) is very memorable for taking a bath. Unfortunately he forgot to get dressed afterwards, in spite of his principles.

From this time onward there was an other interval, the other end point of which was Descartes (1596–1650), who was divinely inspired to invent analytical geometry, and was once found sitting inside a stove to keep himself warm. He also discovered that he existed, and, moreover, he was able to prove it.

Newton (1642–1727) was very memorable indeed, chiefly for having just missed living in St. John's. To console himself he invented the Calculus.

Newton is also memorable for having been admired by Taylor, who invented Maclaurin's series and admired Newton. However, Taylor lived in St. John's and so was luckier than Newton.

The next important mathematician is the Bernoullis. In spite of his having invented numbers, nobody knows how many of him there were, and he lived all over the century. He was called Nicholas, Jacob and John, and one of him was called Daniel.

Euler (1707–1783), Lagrange (1736–1813), and Laplace (1749–1827) are all famous for inventing equations. Only one of Laplace's equations is well known, but this is enough for anyone. It makes electricity and hydrodynamics much easier for people who don't have to solve it. Euler and Lagrange both went about varying things, which caused the calculus of variations. This was both memorable and regrettable.

Gauss invented so many things that it just isn't true. These included the magnetism of the earth, the theory of equations, Cauchy's theorem, and the Cauchy-Riemann equations. In fact, whenever anyone invented anything in the first half of the nineteenth century, Gauss had invented it twenty years earlier, and was still alive to tell him so. He was born in 1777, died in 1855, and lived all the years in between. He was very memorable, and a good thing.

[* *Eureka*, the journal of the Cambridge Archimedeans Society, is always a delight for its freshness and vitality. We are indebted to its Editor and to Messrs. Slessenger and Curtis for permission to reprint this *jeu d'esprit* from *Eureka*, No. 12. Copies of the number can be obtained from the Editor, *Eureka*, The Arts School, Bene't Street, Cambridge, price 1s. 6d. post free.]

† See Sellars and Yeatman, " *1066 And All That*," or any similar standard work.

‡ This includes the year 0.

Cauchy's theorem is very important, but is much harder to prove now than it was when Gauss invented it.

Lobatchewski (1793–1856) must have failed an examination in geometry when he was at school, for he made things harder for everyone by inventing non-Euclidean geometry—just to get his revenge, of course. This was especially bad for the railways, since it made parallel lines so much more difficult.

Hamilton (1805–65) was an Irishman. When he had learnt thirteen languages before he had left school, he decided that there was no future in this, and took up mathematics. He invented Hamilton's principle, the Hamiltonian, the Hamilton-Jacobi theorem, and the Hamilton-Cayley theorem, but not the Hamilton Academicals. Towards the end of his life he also invented quaternions, but nobody except himself ever fell in love with them.

Weierstrass (1815–97) is memorable because of Sonja Kowalewski (1850–91), who, of course, is memorable because of Weierstrass. He said that if you put infinitely many things into a small space, some of them would be pretty close together.

The most memorable of all mathematicians was John Couch Adams (1819–90). He had the good fortune to live in St. John's, and was named after this society. He discovered Neptune just after Leverrier, and would have discovered it before if the Astronomer Royal had kept his eyes open.

Charles Lutwidge Dodgson was a minor Oxford mathematician who must not be confused with Lewis Carroll, whom he impersonated when sending copies of his works to Queen Victoria. They lived contemporaneously.

The chief problem treated by Carroll was that of the Cheshire cat. His treatment is essentially unsound, however, since he says : ". . . this time the cat vanished quite slowly, beginning with the end of the tail, and ending with the grin, which remained some time after the rest of it had gone." * It is obvious that, by the time the tail had disappeared, the cat would be a Manx cat. This is a contradiction, since it was a Cheshire cat, by hypothesis. Carroll also discussed the increased angular velocity of the world if everybody minded his own business.

Riemann (1826–66) invented the tensor calculus, and thus caused the theory of relativity.

In 1895 Bertrand Russell stated the following theorem : the class of all classes which are not members of themselves is either a member of itself or not. Whichever it is, it is the other. This a contradiction, and the end of mathematics.

W. W. O. S. AND A. R. C.

* Lewis Carroll, *Alice's Adventures in Wonderland*.

GLEANINGS FAR AND NEAR.

1633. There was a rather dreadful kind of music which darkened the eighteenth century, and which might be called purely intellectual music ; but to me such music has never justified itself, what is interesting in it being much more completely achieved in the field of pure mathematics.—Filson Young, *Shall I Listen?*, p. 59. [Per Mr. E. J. F. Primrose.]

1634. (Referring to the B.B.C. time signal.) But the synchronization of thought by an instantaneous signal makes time, in a curiously paradoxical way, a reality by abolishing it ; and with it space—that other perhaps arbitrary and illusory dimension—is abolished also.—Filson Young, *Shall I Listen?*, p. 254. [Per Mr. E. J. F. Primrose.]

THE PROBLEM OF THE SWING.

I. By A. Liebetegger.

The fact that a mechanical system so well known to so many in its practical aspect as the swing has been thoroughly neglected in textbooks and problem papers may justify a few remarks on it here.

The only reference to the swing which I have been able to find is the question requiring it to be shown that the amplitude of the swing may be increased by internal action only, if, when the swing passes through its lowest position, the distance of the centre of gravity from the point of suspension is (suddenly) shortened from l to $(l - h)$, the full distance l being restored at the end of the upward swing. Then the angle α in one extreme position may be increased to an angle β in the next extreme position given by the equation

$$\sin \beta = l^3 \sin \alpha/(l - h)^3.$$

While it is not denied that by varying the distance of the centre of gravity between the limits l and $(l - h)$ in the appropriate positions the amplitude may be increased according to the above equation, careful observation of experts on the swing does not confirm that their movements correspond to simply moving their centre of gravity up and down.

Furthermore, it is clear from the equation that, unless $\alpha \neq 0$, β cannot be finite, *i.e.* the swing must be in motion to begin with, before the amplitude can be increased. In other words, the swing cannot be started from rest in the position $\alpha = 0$. All the experts consulted, however, agree that they certainly can start the swing swinging *from rest*.

The mechanical equivalent of their movements, therefore, must amount to something different from merely varying the position of their centre of gravity. If we can describe, in mechanical terms, movements suitable for starting the swing from rest, these movements would presumably account also for the subsequent increase of the amplitude of the swing.

If, in addition, the centre of gravity is moved in the manner mentioned above, this would no doubt help to increase the amplitude, but it is not essential. We therefore assume, in what follows, that the centre of gravity is at a constant distance l from the point of suspension.

Let the swing consist of a body free to rotate about a horizontal axis through its centre of gravity G, this axis being suspended by means of light rods from a parallel fixed axis through O. Let the angle between the plane of the two horizontal axes and the downward vertical through O be θ.

The body, initially at rest with its centre of gravity at A vertically below O, is given *by internal action* a clockwise angular momentum of magnitude H. By the conservation-law of angular momentum G will begin to rotate about O in the anti-clockwise sense (*i.e.* from A towards B) with an angular velocity given by $ml^2\dot\theta = H$. Using the energy principle we easily find that G comes to rest at B, where $\theta = \theta_1$ given by the equation

$$mgl(1 - \cos \theta_1) = H^2/2ml^2. \quad\quad\quad\quad\quad\quad (1)$$

While G is instantaneously at rest at B, the angular momentum is reversed —again by internal action. By the conservation-law of angular momentum the change of angular momentum $2H$ in the anti-clockwise sense must result in an equal angular momentum of G about O in the clockwise sense, so that G begins to move from B towards A with an angular velocity $\dot\theta_1$ given by $ml^2\dot\theta_1 = 2H$. Again using the conservation-law of energy, and equation (1) above, we find that the body comes instantaneously to rest again at C, where $\theta = \theta_2$, which is given by the equation

$$mgl(1 - \cos \theta_2) = 5H^2/2ml^2. \quad\quad\quad\quad\quad\quad (2)$$

While G is instantaneously at rest at C, the angular momentum of the body about G is again reversed. The change of angular momentum $2H$ in the clockwise sense will set up an anti-clockwise angular momentum of G
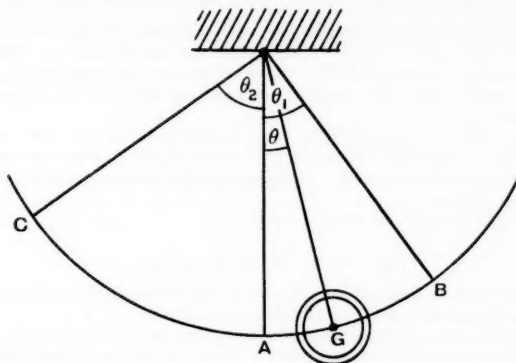
about O, so that G begins to move from C towards A with an angular velocity θ_2 given by $ml^2\theta_2 = 2H$. Using the energy principle again and equation (2) we find the next amplitude θ_3, which is given by the equation

$$mgl(1 - \cos \theta_3) = 9H^2/2ml^2. \quad\quad\quad\quad\quad\quad(3)$$

Writing $2\sin^2 \frac{1}{2}\theta_k$ for $(1 - \cos \theta_k)$ we get the position where the swing comes momentarily to rest for the k-th time given by

$$\theta_k = 2\sin^{-1}\left\{ \frac{H}{2ml^2} \sqrt{\left(\frac{l}{g}\right)} \sqrt{(4k - 3)} \right\} \quad\quad\quad\quad\quad(4)$$

Although by reversing the angular momentum of the body about G every time it comes momentarily to rest we can increase the amplitude, this is, as observation shows, not *quite* the way a swing is operated. A closer approximation to the movements of the human body on a swing is obtained if we work our model swing as follows :

Let the swing start as before and reach the point B, with θ_1 given by equation (1). But at B, instead of reversing the rotation of the body about G, we merely stop it so that the change of angular momentum about G is H in the anti-clockwise sense. Then G begins to move clockwise from B towards A with angular velocity given by $ml^2\theta_1 = H$. When G passes through A the body is suddenly set in rotation again with an anti-clockwise angular momentum H.

By applying (i) the energy equation to the downward swing to find the angular velocity at A ; (ii) the angular momentum equation at A ; (iii) again the energy equation to the upward swing, we find that the body comes to rest at C where $\theta = \theta_2$, given by the equation

$$mgl(1 - \cos \theta_2) = (3 + 2\sqrt{2})H/2ml^2. \quad\quad\quad\quad\quad(5)$$

Stopping the rotation of the body again at C, and again setting it in rotation clockwise when it passes through A, we find, applying the conservation-law (i) of angular momentum at C ; (ii) of energy between C and A ; (iii) of

angular momentum at A ; (iv) of energy to the upward swing from A towards B, that the body comes to instantaneous rest somewhere above B, where $\theta = \theta_3$ is given by

$$mgl(1 - \cos \theta_3) = \{5 + 2\sqrt{2} + 2\sqrt{(4 + 2\sqrt{2})}\}H^2/2ml^2. \quad\quad\quad (6)$$

Comparing (5) and (6) respectively with (2) and (3) we find that, since $5 < 3 + 2\sqrt{2}$, $9 < \{5 + 2\sqrt{2} + 2\sqrt{(4 + 2\sqrt{2})}\}$, that the amplitudes θ_2, θ_3 are greater when the swing is operated so that the body is not rotating during the downward swing, than when the angular momentum about G is reversed at the extreme positions.

The expert on the swing then, it is suggested, carries out movements similar to the movements of the body of our model swing, which are essentially:
  (1) during the upward swing : rotation in the sense opposite to the sense of the swing ;
  (2) during the downward swing : rest.

II. By F. H. NORTHOVER.

The following note generalises a question set in the B.Sc. (General) Applied Mathematics Paper II (1944) of London University.

" A man stands on a swing, and, for the purposes of this question, he may be regarded as a particle whose distance from the smooth horizontal axis of the swing is l when he crouches, and $l - h$ when he stands. As the swing falls the man crouches, and as it rises he stands, the change-over being assumed instantaneous. If the swing falls through an angle α and then rises through an angle β, show that

$$\sin \tfrac{1}{2}\beta = \{l/(l - h)\}^{3/2} \sin \tfrac{1}{2}\alpha."$$

In the idealised case, the energy equations before and after standing up are

$$\tfrac{1}{2}mV^2 = mg(l - h)(1 - \cos \beta),$$
$$\tfrac{1}{2}mv^2 = mgl(1 - \cos \alpha),$$

and, by conservation of angular momentum,

$$lv = (l - h) V.$$

Hence $\quad\quad\quad \sin \tfrac{1}{2}\beta = \{l/(l - h)\}^{3/2} \sin \tfrac{1}{2}\alpha.$

Putting $\beta = \alpha_1$, if the man sits down at α_1, then at this point $v = V$, since there is no change of angular momentum. He gets up suddenly at the lowest point of the return swing. Thus

$$\sin \tfrac{1}{2}\alpha_2 = \{l/(l - h)\}^{3/2} \sin \tfrac{1}{2}\alpha_1.$$

Thence after r swings,

$$\sin \tfrac{1}{2}\alpha_r = \{l/(l - h)\}^{3r/2} \sin \tfrac{1}{2}\alpha.$$

In this idealised case it is possible for the man to increase his angle of swing indefinitely.

The general case.

In practice the man does not get up instantly, but starts from rest and ends at rest in a small but finite time T. It is possible to show that the energy he gives to the system in this action is made larger by making $T \to 0$, or, in practice when we take $T = 0$ as a first approximation.

Let P be the pressure he exerts with his feet, then

$$m(-\ddot{r} + r\dot{\theta}^2) = P - mg \cos \theta, \quad\quad\quad\quad\quad (1)$$
$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -mg \sin \theta. \quad\quad\quad\quad\quad (2)$$

Thus the work done by him on the system is

$$-\int_{l}^{l-h} P\, dr = m\int_{l-h}^{l} (-\ddot{r} + r\dot{\theta}^2 + g\cos\theta)\, dr$$

$$= m\int_{l-h}^{l} (r\dot{\theta}^2 + g\cos\theta)\, dr \qquad \qquad (2a)$$

since $\dot{r} = 0$ at both limits.

Now (2) gives

$$\frac{r^2\dot{\theta}}{l^2\omega} - 1 = -\frac{g}{l^2\omega}\int_{0}^{T} r\sin\theta\, dt. \qquad \qquad (3)$$

If $T \to 0$, this gives the angular momentum relation $r^2\dot{\theta} = l^2\omega$, but (3) shows that, for a finite time T, $r^2\dot{\theta} < l^2\omega$. Thus the work he does, by evaluating (2a), is found to be less than

$$\tfrac{1}{2}ml^2\omega^2\{l^2/(l-h)^2 - 1\} + mgh,$$

that is, less than he would do on getting up instantaneously.

This represents a gain of dynamical energy for the system. There is, however, a reverse effect, leading to a loss of total energy when he sits down at the top of the swing. By exactly similar analysis it is easy to show that this will be as small as possible when the action can be regarded as instantaneous, for this loss is

$$m\int_{l-h}^{l} (r\dot{\theta}^2 + g\cos\theta)\, dr > mgh\cos\beta.$$

We therefore see that the optimum conditions occur when the actions of getting up and sitting down occur so quickly that they can be considered as instantaneous with negligible error. We proceed to investigate the matter along these lines.

Getting up.

With this approximation, then,

$$-\ddot{r} = \left(\frac{P}{m} - g\right) - l^4\omega^2/r^3. \qquad \qquad (4)$$

Suppose that the greatest upward force he can exert with his arms and legs while getting up is nmg; then he has to stop exerting this force at some radial distance r_1, say, in order that he may come to rest at $l - h$.

Integrating (4)

$$-\tfrac{1}{2}\dot{r}_1^2 = (n-1)g(r_1 - l) + \tfrac{1}{2}l^4\omega^2/r_1^2 - \tfrac{1}{2}l^2\omega^2, \qquad \qquad (5)$$

and, for the last part of the motion,

$$\tfrac{1}{2}\dot{r}_1^2 = -g(l - h - r_1) + \tfrac{1}{2}l^4\omega^2/(l-h)^2 - \tfrac{1}{2}l^4\omega^2/r_1^2. \qquad \qquad (6)$$

Thus r_1 is given by

$$0 = -g\{-h + n(l - r_1)\} + \tfrac{1}{2}l^2\omega^2\{l^2/(l-h)^2 - 1\} \qquad \qquad (7)$$

or

$$n(l - r_1) = h + l^2\omega^2\{l^2/(l-h)^2 - 1\}/2g. \qquad \qquad (8)$$

Hence

$$\omega^2\{l^2/(l-h)^2 - 1\} \leqslant (n-1)\,.\,2gh/l^2, \qquad \qquad (9)$$

which shows that the work he does on uprising is less than or equal to $nmgh$.

Sitting down.

Suppose that the greatest force he can exert with his arms in his progression seatwards is kmg; let him have moved x radially so that the retarding force nmg may just bring him to rest on the seat. If the time he takes to sit down can be regarded as zero to a first approximation, the work done on him is equal to the loss of potential energy in sitting down, namely, $mgh \cos \beta$, and the time taken is

$$\sqrt{\left\{ \frac{2h}{g} \cdot \frac{(n+k)}{(n - \cos \beta)(k + \cos \beta)} \right\}} . \qquad \qquad (10)$$

Hence the condition for an overall gain of energy is

$$\tfrac{1}{2} m l^2 \omega^2 \{ l^2/(l-h)^2 - 1 \} + mgh > mgh \cos \beta,$$

which is always satisfied.

There are, however, two limiting factors to an indefinite increase of angle : (i) the condition (9) above; (ii) the requirement that the times of uprising and downsitting shall be sufficiently small. We shall investigate (ii) in the following sections, and find that it leads to a condition similar to (i) but more stringent.

On the validity of the approximations.

When the man gets up, then from (3) we have

$$\theta < l^2 \omega/r^2, \quad \text{so that} \quad \theta < l^2 \omega t/r^2.$$

Hence

$$\int_0^\tau r \sin \theta \, dt < l^2 \omega \int_0^\tau (t/r) \, dt < \tfrac{1}{2} l^2 \omega t^2/r.$$

Thus

$$\theta < \frac{l^2 \omega}{r^2} \left(1 - \frac{gt^2}{2r} \right).$$

Hence in order that the assumption of instantaneous uprising may give a good approximation it is sufficient that T is small compared with $\sqrt{\{2(l-h)/g\}}$, which we write

$$T \ll \sqrt{\{2(l-h)/g\}}.$$

Also we must have

$$T \ll (l-h)^2/l^2 \omega,$$

since also $\theta \ll 1$.

Now when he sits down

$$r^2 \dot{\theta} = -g \int_0^t r \sin \theta \, dt',$$

so that

$$| \dot{\theta}_t | < \frac{gr}{r^2} \int_0^t \sin \beta \, dt'$$
$$< (gt/r) . \sin \beta.$$

Thus

$$| \theta_t | < \{ gt^2/2(l-h) \} . \sin \beta.$$

Hence if T' be the time for sitting down,

$$| \theta_{\tau'} | < \{ gT'^2/2(l-h) \} . \sin \beta.$$

This angle is small provided that

$$T' \ll \sqrt{\{2(l-h)/g \sin \beta\}}.$$

But T' is given by (10). Thus when β becomes sizeable we must have $h \ll l$.

We now have to examine the case of his uprising, and to do this we shall

require the time of uprising T. This requires elliptic integrals, so we shall show that the above condition $h \ll l$ is sufficient for the validity of our approximation when he gets up, by obtaining an upper bound for T.

Now, for the motion in (l, r_1),

$$\dot{r}^2 = 2\{(n-1)g - \tfrac{1}{2}l^2\omega^2(l+r)/r^2\}(l-r) = F(r), \text{ say,}$$

and in $(r_1, l-h)$

$$\dot{r}^2 = 2\{g + \tfrac{1}{2}l^4\omega^2(r+l-h)/r^2(l-h)^2\}\{r-(l-h)\} = G(r), \text{ say.}$$

Hence

$$T = \int_{l-h}^{r_1} \frac{dr}{\sqrt{G(r)}} + \int_{r_1}^{l} \frac{dr}{\sqrt{F(r)}} \cdot$$

The first integral is less than $\sqrt{(2h/g)}$ and the second is less than

$$\sqrt{\left\{ \frac{2(l-r_1)}{(n-1)g - \tfrac{1}{2}l^2\omega^2(l+r_1)/r_1^2} \right\}} ,$$

and since $h \ll l$, this is less than

$$\sqrt{\left\{ \frac{2h}{(n-1)g - l\omega^2} \right\}} , \text{ approximately, } \quad \ldots\ldots(11)$$

and our previous condition on ω becomes

$$l\omega^2 < (n-1)g.$$

But if $l\omega^2$ becomes so close to $(n-1)g$ that the expression in (11) ceases to be of order $\sqrt{(2k/g)}$ the writer has shown that in this case T becomes of the order $\sqrt{(l/g)}$; and this would then invalidate the condition $T \ll \sqrt{\{2(l-h)/g\}}$. Hence to satisfy *all* the conditions on T and T' we must have $h \ll l$, *and*

$$(n-1)g - l\omega^2$$

of order g.

We shall therefore take

$$(n-1)g - l\omega^2 > g$$

or

$$l\omega^2 < (n-2)g$$

with

$$h \ll l.$$

To this order of approximation, therefore, the angle $\beta_{\max}$ is determined by

$$1 - \cos\beta_{\max} = \tfrac{1}{2}l(\omega_{\max})^2$$

or

$$\cos\beta_{\max} = 1 - \tfrac{1}{2}(n-2),$$

so that

$$\beta_{\max} = \cos^{-1}(2 - \tfrac{1}{2}n).$$

As n is probably less than 3 this shows that, for straightforward getting up and sitting down, the maximum angle through which the man can swing himself is about $60°$.

III. By B. Thwaites.

It is an easily observable fact that a child on its swing in the garden, by maintaining or increasing the amplitude of the swing's oscillations, is doing useful work in some way which may exceed work done by the various frictional forces. A simple analytical explanation of this phenomenon has not been given to my knowledge, and certain solutions are given below of the equations of a motion which is a simplified form of the child on the swing.

Before the analysis is set out, it is necessary to obtain an idea of the fundamental effect a child produces when it changes the position of its body on the

seat of the swing. If we regard the child and swing as a complete system, to the work done by external forces such as air resistance must correspond useful work done by the child, and this useful work can only be done by a force acting along the ropes and through the pivot upon the child which results in a movement up the ropes of the centre of gravity of the child. Thus we can see at once a possible technique of swinging : the child as it swings past the lowest point raises its centre of gravity up the ropes until the swing reaches its greatest amplitude, then lets itself fall freely until restrained by the full length of the swing ropes, then swings back normally to the lowest point. The work the child has done in climbing up the ropes in the frictionless case will result in an increase of angular velocity on the return to the lowest point, and in the frictional case will counterbalance the work done by external frictional forces.
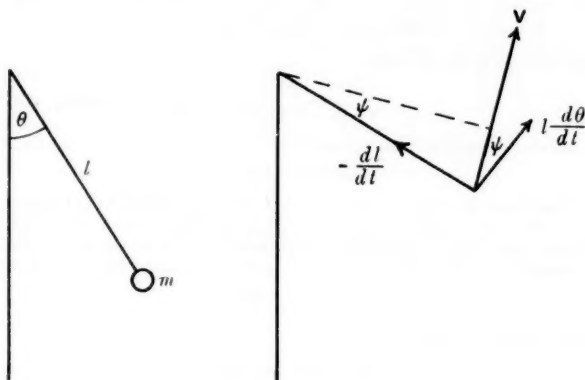

FIG. 1.

$$V = \sqrt{\left\{ \left(\frac{dl}{dt}\right)^2 + l^2 \left(\frac{d\theta}{dt}\right) \right\}^2}.$$

With this technique in mind, we make the idealised frictionless case of a mass m swinging on a rope length l_0 attached to a pivot. At time t the angular displacement of the rope from the vertical is θ and the distance of the mass from the pivot is l ($l \leqslant l_0$). We assume $dl/dt = 0$ at $\theta = 0$.

The equation of motion obtained by equating rate of change of angular momentum to the moment of the forces is

$$\frac{d}{dt}\left(l^2 \frac{d\theta}{dt}\right) = -gl \sin \theta$$

or

$$2\frac{dl}{dt}\frac{d\theta}{dt} + l\frac{d^2\theta}{dt^2} + g \sin \theta = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Integrating this equation with respect to θ from $\theta = \alpha$, α being the maximum value of θ where $d\theta/dt = 0$, we get

$$l\left(\frac{d\theta}{dt}\right)^2 - 2g(\cos\theta - \cos\alpha) + 3\int_\alpha^\theta \frac{dl}{d\theta}\left(\frac{d\theta}{dt}\right)^2 d\theta = 0. \quad \ldots\ldots\ldots(2)$$

In the case $l = \text{constant} = l_0$, the integral term vanishes identically and we get the well-known equations

$$\left(l_0 \frac{d\theta}{dt}\right)^2 = v^2 = 2gl_0 \left(\cos\theta - \cos\alpha\right)$$

and

$$(t_\alpha - t)\sqrt{\frac{g}{l_0}} = \int_\theta^\alpha \frac{d\theta}{\sqrt{\{2\left(\cos\theta - \cos\alpha\right)\}}} \; .$$

To proceed with (2), put

$$\frac{dl}{d\theta}\left(\frac{d\theta}{dt}\right)^2 = gf(\theta), \quad f(\alpha) = 0, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and an integration gives

$$\log l/l_\alpha = -\int_\theta^\alpha \frac{f(\theta)\,d\theta}{2\left(\cos\theta - \cos\alpha\right) + 3F(\theta)} = \phi(\theta), \quad\ldots\ldots\ldots\ldots(4)$$

where $F(\theta) = \int_0^\alpha f(\theta)\,d\theta$. In conjunction with (3) this gives

$$(t_\alpha - t)\sqrt{\frac{g}{l_\alpha}} = \int_\theta^\alpha \left\{\frac{e^{\phi(\theta)}}{2\left(\cos\theta - \cos\alpha\right) + 3F(\theta)}\right\}^{1/2} d\theta \quad\ldots\ldots\ldots\ldots(5)$$

and

$$v_0^2 = l_0 \left(\frac{d\theta}{dt}\right)^2_{\theta=0} = 2l_0 g\left\{(1 - \cos\alpha) + \frac{3}{2}\int_0^\alpha f(\theta)\,d\theta\right\} \quad\ldots\ldots\ldots\ldots(6)$$

For any assumed $f(\theta)$, (4) and (5) give l and t in terms of θ; (6) shows clearly that if the choice of $f(\theta)$ for ascent differs from that for the descent (regarding l_0, the value of l at $\theta = 0$ as fixed and providing the downward acceleration never exceeds g) in general the velocities at $\theta = 0$ will differ. The principle of swinging has therefore been established, and we content ourselves with one or two examples.

(i) *Mass falling freely under gravity.* It is easy to obtain directly that in this case, when $l_\alpha \sin\alpha = l\sin\theta$ and $d^2(l\cos\theta)/dt^2 = g$,

$$f(\theta) = -2\sin\theta\cos\theta\sin\left(\alpha - \theta\right)\operatorname{cosec}\alpha.$$

Hence

$$F(\theta) = \int_\theta^\alpha f(\theta)\,d\theta = -\tfrac{2}{3}\operatorname{cosec}\alpha\left\{\sin\alpha\left(\cos^3\theta - \cos^3\alpha\right) + \cos\alpha\left(\sin^3\theta - \sin^3\alpha\right)\right\}$$

and (4) gives, after some simplification,

$$\log l/l_\alpha = -\int_\theta^\alpha \frac{\sin\theta\cos\theta\sin\left(\alpha - \theta\right)d\theta}{\sin^2\theta\sin\left(\alpha - \theta\right)} = \log\left|\frac{\sin\alpha}{\sin\theta}\right|,$$

or $l_\alpha \sin\alpha = l\sin\theta$, an initial condition which is verified by the full analysis.

(ii) *A certain set of solutions* which illustrate the principle well are given by

$$f(\theta) = k_n \sin\theta\left(\cos\theta - \cos\alpha\right)^n,$$

so that

$$F(\theta) = \int_\theta^\alpha f(\theta)\,d\theta$$
$$= k_n\left(\cos\theta - \cos\alpha\right)^{n+1}/(n + 1).$$

Then (4) gives

$$\log l/l_\alpha = -\int_\theta^\alpha \frac{k_n \sin\theta\left(\cos\theta - \cos\alpha\right)^n d\theta}{2\left\{\left(\cos\theta - \cos\alpha\right) + 3k_n\left(\cos\theta - \cos\alpha\right)^{n+1}/(n + 1)\right\}}$$
$$= -\frac{n+1}{3n}\log\left|1 + 3k_n\left(\cos\theta - \cos\alpha\right)^n/(n + 1)\right|.$$

Thus $$l=l_\alpha\{1+3k_n\,(\cos\theta-\cos\alpha)^n/(n+1)\}^{-(n+1)/3n}. \qquad (7)$$

If the swinging technique is to swing according to one n upwards and another downwards, k_n must be chosen so that l_0/l_α is independent of n and substitution of this value of k_n into (7) gives

$$l=l_\alpha\left[1+\{(l_0/l_\alpha)^{-3n/(n+1)}-1\}\left\{\frac{\cos\theta-\cos\alpha}{1-\cos\alpha}\right\}^n\right]^{-(n+1)/3n} \qquad (8)$$
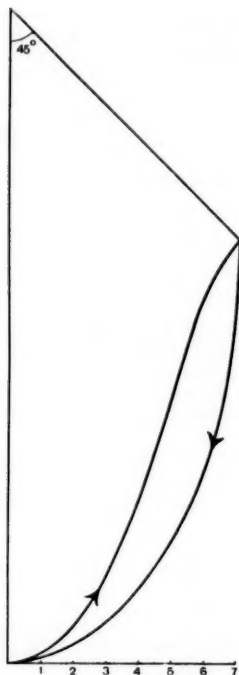

FIG. 2.

Path taken in the case $l_0=2l_\alpha$, $\alpha=45°$, $n=1,2$.
Ratio of velocities at lowest point is $1\cdot188$.

From (6) it is found that

$$v_0{}^2/2l_0g=(1-\cos\alpha)(l_0/l_\alpha)^{-3n/(n+1)}. \qquad (9)$$

In practice, a child does not raise itself far up the swing ropes, so that putting $l_\alpha=l_0(1-\delta)$, (8) and (9) become, to $O(\delta^2)$,

$$l=l_\alpha\left\{1+\delta\left(\frac{\cos\theta-\cos\alpha}{1-\cos\alpha}\right)^n\right\}$$
$$v_0{}^2/2l_0g=(1-\cos\alpha)\left(1+\frac{3n}{n+1}\delta\right) \qquad (10)$$

For a gain in energy, therefore, the value of n on the ascent must be greater than its value on the descent. It is simple to verify that the increase of potential energy at $\theta = 0$ exactly equals the work done on the swing. The physical rope condition that the downward acceleration must not exceed g is rather complicated analytically in this case, and, of course, is not necessary if the ropes are replaced by light bars capable of bearing a compressive force.

As a numerical example, we can take the rather extreme case $l_0/l_\alpha = 2$, $\alpha = 45°$, and $n = 1$ or 2.

For $n = 1$, $\qquad l/l_\alpha = \{1 - 2 \cdot 207\ (\cos \theta - \tfrac{1}{2}\sqrt{2})\}^{-2/3}$;

for $n = 2$, $\qquad l/l_\alpha = \{1 - 8 \cdot 742\ (\cos \theta - \tfrac{1}{2}\sqrt{2})^3\}^{-1/2}$.

The following table can be quickly compiled, x and y being rectangular coordinates, origin at the pivot, and the suffixes u and d denoting the up and down swings respectively.

θ	$\cos\theta - \cos 45°$	$n = 2$			$n = 1$		
		l/l_α	x_u/l_α	y_u/l_α	l/l_α	x_d/l_α	y_d/l_α
0	0·2929	2	0	2·0	2	0	2·0
9°	0·2806	1·791	·280	1·769	1·903	·298	1·880
18°	0·2440	1·444	·446	1·373	1·675	·518	1·593
27°	0·1839	1·192	·541	1·062	1·415	·542	1·261
36°	0·1019	1·049	·617	·849	1·185	·696	·959
45°	0	1·000	·707	·707	1·000	·707	·707
$v_0{}^2/2gl_0(1 - \cos \alpha)$		0·25			0·3536		

Figure 2 shows the path of the mass in this case, which confirms the original guess at the technique given in the second paragraph.

Various problems immediately suggest themselves as a consequence of the above analysis. Probably one of the most interesting is this : given l_0, l_α, α, and starting at $\theta = \alpha$, what path of descent gives the maximum value of $d\theta/dt$ at $\theta = 0$ where $dl/d\theta = 0$. One could hardly expect, however, that a child would take this most advantageous path on its swing, and therefore we remain content at this point with having demonstrated that it is in fact possible to swing a swing.

1635. Book publishing is a personal affair and whereas in the amalgamation of many businesses $1 + 1 = 2 + x$, in book publishing $1 + 1$ would equal $2 - x$.—Sir Stanley Unwin, letter in the *Economist*, October 15, 1949. [Per Mr. M. Bridger.]

1636. The highest common factor of semantic content in appropriate subjects of *all* verbs is zero.—L. Hogben, *Interglossa*, p. 42. [Per Prof. M. H. A. Newman.]

1637. Very late in life, when he was studying geometry, someone said to Lacydes, " Is it then a time for you to be learning now? " " If it is not ", he replied, " when will it be? "—Diogenes Laertius, quoted in Bartlett's *Familiar Quotations*, 11th ed., p. 1014. [Per Mr. F. Bowman.]

1638. *The Cambridge Journal* is a monthly review, which will consider matters of permanent and temporary interest in the fields of literature, history, economics, psychology, philosophy and politics, as well as science in so far as it concerns the educated man. [Per Dr. W. Edwards Deming.]

A SIMPLE TEST FOR THE REALITY AND SIGN OF THE ROOTS OF TWO DETERMINANTAL EQUATIONS OF HIGH DEGREE.

By M. J. MOORE.

TWICE in the course of some work on chemical kinetics * it became necessary to determine the type of the solution to a generalised set of any number of first order simultaneous differential equations. The problem reduced in each case to a question of the reality and sign of the roots of the auxiliary equation, which was, of course, determinantal in form. The required information could be deduced from long-established theorems on the roots of symmetric and skew-symmetric determinantal equations, but the following proofs, which are on a much simpler level, may be of interest.

The method is by induction on a series of similar determinantal equations formed from the first m rows and columns of the original determinant. The principle involved is that if two functions, $F(x)$ and $G(x)$, are both positive or both negative for $a < x < b$, and one changes sign at b, the other at $c\ (\geqslant b)$, while neither changes sign a second time for $b \leqslant x \leqslant c$, then the sum function $[pF(x) + qG(x)]$, where p, q are real constants > 0, changes sign in the interval $b < x < c$ (or at b if $b = c$).

Definition. If $F(x) = 0$ has m real roots, $f_1, f_2 \ldots f_m$, and $G(x) = 0$ has m or $m - 1$ real roots $g_1, g_2 \ldots g_{m-1}, (g_m)$, these roots are here said to be *alternate* if $f_1 \leqslant g_1 \leqslant f_2 \leqslant \ldots \leqslant g_{m-1} \leqslant f_m\ (\leqslant g_m)$ and to be *strictly alternate* if

$$f_1 < g_1 < f_2 < \ldots < g_{m-1} < f_m\ (< g_m).$$

It will always be assumed that if the roots of an equation are numbered, they are numbered in order of size starting with the lowest.

EQUATION I.

$$\Delta_n(x) \equiv \begin{vmatrix} x & a_2 & 0 & \ldots & 0 & 0 \\ -c_1 & x & a_3 & \ldots & 0 & 0 \\ 0 & -c_2 & x & \ldots & 0 & 0 \\ 0 & 0 & -c_3 & \ldots & 0 & 0 \\ \multicolumn{6}{c}{\dotfill} \\ 0 & 0 & 0 & \ldots & x & a_n \\ 0 & 0 & 0 & \ldots & -c_{n-1} & x \end{vmatrix} = 0 \text{ where } a_j > 0,\ c_j > 0.$$

It is to be proved that if n is even the equation has n pure imaginary roots, and if n is odd one root is zero, $(n - 1)$ are purely imaginary. Let $\Delta_m(x)$, where $2 \leqslant m \leqslant n$, be the determinant formed from the first m rows and columns of $\Delta(x)$.

$$\Delta_m(x) = \begin{vmatrix} x & a_2 & 0 & \ldots & 0 & 0 \\ -c_1 & x & a_3 & \ldots & 0 & 0 \\ 0 & -c_2 & x & \ldots & 0 & 0 \\ 0 & 0 & -c_3 & \ldots & 0 & 0 \\ \multicolumn{6}{c}{\dotfill} \\ 0 & 0 & 0 & \ldots & x & a_m \\ 0 & 0 & 0 & \ldots & -c_{m-1} & x \end{vmatrix}$$

By examination of the determinant, it becomes obvious that $\Delta_m(x)$ is a polynomial of degree m in x such that

* (Equation I.). Margaret J. Moore, *Trans. Faraday Soc.*, XLV, 12, 1098, 1949. (Equation II.). K. G. Denbigh, M. J. Hicks (Moore), F. M. Page, *Trans. Faraday Soc.*, XLIV, 7, 479, 1948.

(1) the coefficient of x^m is $+1$;

(2) x^{m-2}, x^{m-4} and alternate powers of x have coefficients >0 ;

(3) x^{m-1}, x^{m-3} and alternate powers of x have zero coefficients.

Then polynomials F_r, G_r, of degree r, with positive coefficients, may be defined so that

$$\Delta_{2r}(x) = F_r(x^2) \quad \text{and} \quad \Delta_{2r+1}(x) = xG_r(x^2).$$

Also

$$\Delta_m(x) = x\Delta_{m-1}(x) + c_{m-1}a_m\Delta_{m-2}(x), \quad \dots\dots\dots\dots\dots\dots\dots(1)$$

so that

$$F_r(x^2) = x^2 G_{r-1}(x^2) + c_{2r-1}a_{2r}F_{r-1}(x^2),$$

$$G_r(x^2) = F_r(x^2) + c_{2r}a_{2r+1}G_{r-1}(x^2).$$

i.e.

$$F_r(y) = yG_{r-1}(y) + c_{2r-1}a_{2r}F_{r-1}(y), \quad \dots\dots\dots\dots\dots\dots\dots(2)$$

$$G_r(y) = F_r(y) + c_{2r}a_{2r+1}G_{r-1}(y). \quad \dots\dots\dots\dots\dots\dots\dots\dots(3)$$

Assume that $F_{r-1}(y) = 0$ and $G_{r-1}(y) = 0$ have each $(r-1)$ real roots, which are respectively $f_1, f_2 \dots f_{r-1}$ and $g_1, g_2 \dots g_{r-1}$, and that

$$g_1 < f_1 < g_2 < \dots < g_{r-1} < f_{r-1} < 0. \quad \dots\dots\dots\dots\dots\dots\dots(4)$$

Consider the functions $yG_{r-1}(y)$, $F_{r-1}(y)$ and $F_r(y)$, which last is the sum of positive constant multiples of the other two. They are all polynomials, of degree r, $r-1$, r respectively in y, and all their coefficients >0 except the zero constant term of $yG_{r-1}(y)$. From (4), the roots of $yG_{r-1}(y) = 0$ and $F_{r-1}(y) = 0$ are strictly alternate.

For y very small $(-y$ large$)$ $yG_{r-1}(y)$ and $F_r(y)$ have the same sign, unlike the sign of $F_{r-1}(y)$. If y is now increased up to g_1, $F_{r-1}(y)$ does not change sign ; therefore, as $yG_{r-1}(y)$ vanishes at g_1, $F_r(y)$ must there take the same sign as $F_{r-1}(y)$ and must, in fact, have changed sign for some value of $y < g_1$. In the open interval (g_1, f_1) the three functions have the same sign, while $f_{r-1}(y)$ changes sign at f_1, $yG_{r-1}(y)$ changes sign at $g_2 (>f_1)$. Therefore $F_r(y)$ changes sign in the open interval (f_1, g_2). By similar reasoning it can be proved that $F_r(y)$ changes sign in every successive interval typified by (f_j, g_{j+1}) up to (f_{r-2}, g_{r-1}), and finally in the open interval $(f_{r-1}, 0)$.

Thus $F_r(y) = 0$ has r real roots which are <0 and strictly alternate with those of $G_{r-1}(y) = 0$.

Similarly, if the same reasoning is applied to $G_r(y)$ considered as the sum of positive constant multiples of $F_r(y)$ and $G_{r-1}(y)$, it is found that $G_r = 0$ has r real roots, all <0 and strictly alternate with those of $F_r = 0$, the smallest being a root of $G_r = 0$.

Thus if the original assumption is true for any small value of r, it is true for any subsequent value of r : and $F_r = 0$, $G_r = 0$ have each r real, distinct roots <0, up to $F_{\frac{1}{2}n} = 0$ if n is is even or $G_{\frac{1}{2}(n-1)} = 0$ if n is odd.

But

$$F_1(x^2) \equiv \Delta_2(x) \equiv x^2 + c_1 a_2.$$

$$G_1(x^2) \equiv \frac{1}{x}\Delta_3(x) \equiv x^2 + (c_1 a_2 + c_2 a_3).$$

Therefore $F_1(y) \equiv y + c_1 a_2$; $F_1(y) = 0$ has one real root, $-c_1 a_2$.

$$G_1(y) \equiv y + (c_1 a_2 + c_2 a_3) \; ; \quad G_1(y) = 0 \text{ has one real root,}$$

$$-(c_1 a_2 + c_2 a_3) < -c_1 a_2 < 0.$$

Therefore $F_2(y) = 0$ has 2 real roots <0 and strictly alternate with the root of $G_1(y) = 0$.

And so on.

Therefore if n is even, $F_{\frac{1}{2}n}(y) = 0$ has $\frac{1}{2}n$ distinct, real roots < 0, so that

$$\Delta_n(x) \equiv F_{\frac{1}{2}n}(x^2) = 0$$

has n distinct pure imaginary roots.

If n is odd, $G_{\frac{1}{2}(n-1)}(y) = 0$ has $\frac{1}{2}(n-1)$ distinct, real roots < 0, so that

$$\Delta_n(x) \equiv xG_{\frac{1}{2}(n-1)}(x^2) = 0$$

has $(n-1)$ distinct pure imaginary roots and one zero root.

EQUATION II.

$$\Delta_n(x) \equiv \begin{vmatrix} x+b_1 & a_2 & 0 & \ldots & 0 & 0 \\ c_1 & x+b_2 & a_3 & \ldots & 0 & 0 \\ 0 & c_2 & x+b_3 & \ldots & 0 & 0 \\ 0 & 0 & c_3 & \ldots & 0 & 0 \\ \multicolumn{6}{c}{\dotfill} \\ 0 & 0 & 0 & \ldots & x+b_{n-1} & a_n \\ 0 & 0 & 0 & \ldots & c_{n-1} & x+b_n \end{vmatrix} = 0, \text{ where } a_j > 0, \; c_j > 0.$$

It is to be proved that this equation has n real, distinct roots. Put $\Delta_m(x)$ for the determinant formed from the first m rows and columns of $\Delta_n(x)$.

Also let
$$P_m(x) \equiv (x+b_m)\,\Delta_{m-1}(x),$$
$$Q_{m-2}(x) \equiv -c_{m-1}a_m\Delta_{m-2}(x).$$

Then
$$\Delta_m \equiv P_m + Q_{m-2}.$$

Suppose that $\Delta_{m-1} = 0$ has $(m-1)$ real roots, $\mu_1, \mu_2, \ldots \mu_{m-1}$, and $\Delta_{m-2} = 0$ has $(m-2)$ real roots, $\lambda_1, \lambda_2, \ldots \lambda_{m-2}$. Suppose also that these two sets of roots are strictly alternate.

Then $P_m = 0$ has m real roots, $\mu_1, \ldots \mu_{m-1}$ and also $-b_m$.

$Q_{m-2} = 0$ has the same $(m-2)$ real roots as $\Delta_{m-2} = 0$.

$(-b_m)$ can occur in any position relative to the other roots, and may even coincide with any λ or any μ.

Put all these roots in pairs in order of size, with the convention that if $-b_m$ coincides with any other root it shall be placed to the right of it. Then there will be $(m-1)$ such pairs, falling into three classes as follows :

(a) From 0 to $(m-2)$ pairs of roots both $\leqslant -b_m$, typified by (μ_i, λ_i).

(b) One pair of roots containing $-b_m$, which is either $(-b_m, \mu_k)$ or $(\mu_k, -b_m)$.

(c) From $(m-2)$ to 0 pairs of roots both $> -b_m$, typified by (λ_{j-1}, μ_j).

Thus (i) Every " pair " contains one μ and only one.

(ii) The second member of one " pair " and the first member of the next comprise a root of each of equations $P_m = 0$, $Q_{m-2} = 0$.

(iii) The first and last roots of all are roots of $P_m = 0$.

Also:

(iv) $\Delta_m, P_m > 0$, $Q_{m-2} < 0$ as x gets very big.

Δ_m, P_m have one sign, Q_{m-2} the other, as x gets very small $[\to -\infty]$.

From (iii), and since $\Delta_m \equiv P_m + Q_{m-2}$, Δ_m takes the sign of Q_{m-2} at the first and last of the whole series of roots.

Therefore and from (iv), Δ_m changes sign for some value of $x < \mu_1$, and for some value of $x > \mu_{m-1}$.

Also, in the interval formed by the first " pair ", Δ_m, P_m and Q_{m-2} have the same sign.

Therefore and from (ii), Δ_m vanishes in the interval between the first " pair " and the second.

Similarly, the three functions have the same sign in every interval formed by a " pair ", and Δ_m changes sign in the interval between any " pair " and the next. (See Fig. 1.)
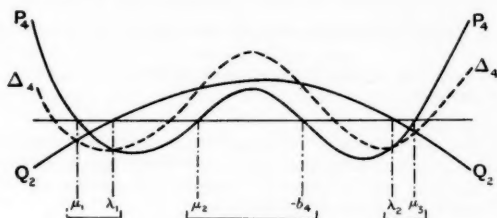

FIG. 1.

If $\mu_k = -b_m$, the interval $(\mu_k, -b_m)$ reduces to a point, where $P_m = 0$, and Δ_m, Q_{m-2} have the same sign, but Δ_m still changes sign between " pairs ". (See Fig. 2.) If $\lambda_{k-1} = -b_m$, Δ_m still vanishes " between " λ_{k-2} and $(-b_m)$,
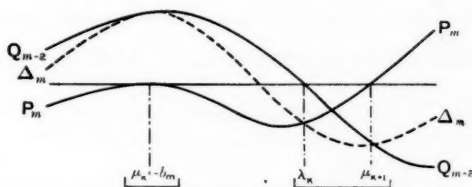

FIG. 2.

i.e. at $\lambda_{k-1} = -b_m$. (See Fig. 3.) In no case can Δ_m vanish at any of the points μ.
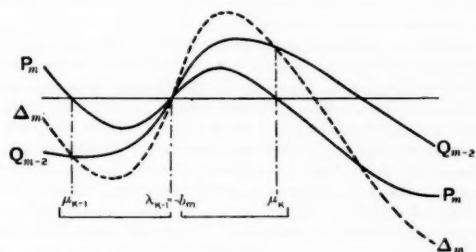

FIG. 3.

Thus $\Delta_m = 0$ has m real roots, which are alternate with the " pairs " defined above, and (from (i)) strictly alternate with the roots, μ_i, of $\Delta_{m-1} = 0$.

Therefore provided that $\Delta_{m-1} = 0$, $\Delta_{m-2} = 0$ have real, strictly alternate roots, $\Delta_m = 0$ also has real roots which are strictly alternate with those of $\Delta_{m-1} = 0$.

G

$$\Delta_1 \equiv x + b_1.$$

$\Delta_1 = 0$ has one real root, $-b_1$.

$$\Delta_2 \equiv (x + b_1)(x + b_2) - a_2 c_1.$$

$\Delta_2 = 0$ has 2 real roots, one $< -b_1$, the other $> -b_1$, i.e. they are strictly alternate with the root of $\Delta_1 = 0$.

Therefore $\Delta_3 = 0$ has **3** real roots strictly alternate with those of $\Delta_2 = 0$. And so on.

Therefore $\Delta_n = 0$ has n real roots, strictly alternate with those of $\Delta_{n-1} = 0$, and therefore distinct.

A. More general case.

If $a_j \geqslant 0$, $c_j \geqslant 0$ only, i.e. sign $>$ replaced by $\geqslant$, it is impossible to prove the *strict* alternation, but otherwise the proof holds (again taking coincident points as the limiting case of adjacent points). Therefore $\Delta_n = 0$ has n real roots, which need not, however, be distinct.

B. Additional conditions.

If $\Delta_m(0) > 0$ for every m, it can be proved that every root of $\Delta_m = 0$, for every m, < 0. (By assuming that this is true for $(m-1)$ and $(m-2)$, and by considering the signs of Δ_m, P_m and Q_{m-2}, at $x = 0$ instead of as $x \to \infty$.)

Thus if $\Delta_m(0) > 0$, $\Delta_n(x) = 0$ has n real roots < 0.

If $\Delta_m(0) \geqslant 0$ for every m, the roots of $\Delta_m(x) = 0$ and $\Delta_n(x) = 0$ are all real and $\leqslant 0$.

If $\Delta_m(0) = (-1)^m D_m$ where $D_m > 0$, it can be proved that the roots of $\Delta_n = 0$ are all > 0.

Sufficient conditions for $\Delta_m(0) \geqslant 0$ or $\Delta_m(0) > 0$ can be found fairly easily by induction. They are :

For $\Delta_m(0) \geqslant 0$. $a_j \geqslant 0$, $c_j \geqslant 0$, $b_j \geqslant a_j + c_j$ for every j.

For $\Delta_m(0) > 0$.   These conditions, and in addition :

(1) $b_j > 0$ for every j.

(2) $b_j > a_j + c_j$ for some value or values of j (taking $a_1 = c_n = 0$).

(3) If $b_{j+1} = a_{j+1} > 0$ then $b_j > c_j$ (i.e. $a_j > 0$ or $b_j > a_j + c_j$).

     If $b_{j-1} = c_{j-1} > 0$ then $b_j > a_j$ (i.e. $c_j > 0$ or $b_j > a_j + c_j$).

(This implies, since $a_1 = 0$, $c_n = 0$, that (unless $b_1 > c_1$), $c_1 > 0$, $c_2 > 0$, and so on until a value, j, is reached such that $b_j > a_j + c_j$. Likewise, after the last value of j for which $b_j > a_j + c_j$, $a_n > 0$, $a_{n-1} > 0$, etc.)   MARGARET J. MOORE.

1639 Lord Baldwin, politics apart, had a pretty wit and a talent for phrase-making. In a recent interview (writes " K.") with two lively writers, Hesketh Pearson and Hugh Kingsmill, the former (as Shaw's biographer) asked Lord Baldwin if he had ever met Mr. Bernard Shaw. Lord Baldwin said that he had. Shaw had sought an interview with him in some connection which he had by now forgotten, and so interesting was his conversation that the Prime Minister (as he then was) had kept him talking for a whole hour. Lord Baldwin added : " Shaw is charming with one man, fidgety with two, and stands on his head for four." It seems a pity that he did not continue with this attractive arithmetical progression. It would have been interesting to learn how Mr. Shaw reacts to eight, sixteen, and up to the millions represented by a modern radio audience.—*Manchester Guardian*. [Per Mr. C. B. Gordon.]

1640. Music of this class is transcendent.

It is superior to ordinary affairs—a kind of international Highest Common Denominator of beauty and achievement.

Until the other day, great composers and artists have loved it for what it is.—*Daily Mirror*, March 29, 1949. [Per Mr. B. D. Price.]

THE PROBABILITY OF A GIVEN ERROR
BEING EXCEEDED IN APPROXIMATE COMPUTATION.*

By S. Inman.

Addition.

I will illustrate the kind of problem I am going to discuss by showing how it applies to the case of addition. Suppose we add a series of numbers, each correct to the nearest unit ; this may be tenths, hundredths or any other unit. The maximum error in the sum of n such numbers is $\frac{1}{2}n$ units. The argument, almost universally applied, is that as the maximum error is $\frac{1}{2}n$ units, the answer is unreliable to that extent. This is bad logic. Let us take an analogy from electricity. When a current is switched on the maximum current is reached only after a time which is infinity. Actually, after a very short time, the difference of the current from the maximum is negligible. Likewise to argue that the sum of n items is unreliable to the extent of $\frac{1}{2}n$ units is not merely bad logic, it is completely wide of the truth.

This problem was investigated by G. J. Lidstone (*Transactions of the Faculty of Actuaries in Scotland*, Vol. XVII, Part 2, No. 157, 1938). Assuming that all errors between $\pm(\frac{1}{2}$ a unit) are equally probable in every number to be added, Lidstone showed that as n increases the probability that the total error will exceed $k\sqrt{n}$ units approaches the value of $1 - \sqrt{\frac{2}{\pi}} \int_0^\lambda e^{-x^2/2} \, dx$ where $\lambda = 3 \cdot 46k$. The implication of this is seen from the probabilities for the following values of k :

Value of k - -	0·195	0·673	0·746	0·950	1·0	1·277	$1\frac{1}{4}$
Proby. total error exceeds $k\sqrt{n}$ units	1/2	1/50	1/100	·1/1000	1/1900	1/100000	1/50000

The probability values converge to zero with great rapidity. To illustrate, suppose we add 100 numbers each correct to the nearest unit. The maximum error is 50 units, but actually there is an even chance that the error will not exceed 2 units ; to suppose that the error is more than 12 units is absurd.

If the average of the numbers be taken instead of their total, the error in the average is $1/n$th of that in the sum, so that the above figures give the probabilities that the error of the average exceeds $k/\sqrt{n}$ units. Thus, in the case of 100 numbers, it is extremely unlikely that the average will be in error by more than 0·1 units. If a ruler is graduated to tenths of an inch anyone can, by estimation, read to 0·01 of an inch. If the ruler is also graduated in cm. and it is used to measure a line in inches and cm. we can obtain, by division, the number of cm. in an inch and it would be reasonable to expect the answer to be correct to 3 significant figures. What this theory says is this. If the experiment is done 100 times, the error in the average is reliably correct to 4 significant figures and not 3 figures. I tested this by drawing 75 lines and the average of my answers was 2·5403 (the correct answer is 2·5400). The value of taking an average is thus seen in a new light. The taking of an average does not merely balance out errors, it has also the effect of getting closer and closer to the correct answer. I have not time to pursue this matter further, but it is necessary to point out that if the ruler or apparatus is inaccurate, its inaccuracy is superimposed on the answer. If

anyone wishes to repeat this experiment he should avoid the use of a wooden ruler. I advise a good steel ruler.

Multiplication and division of two numbers each correct to n figures.

The main part of my paper deals with multiplication and division and, from this point, apart from a few introductory remarks, the matter with which I am dealing has not, as far as I know, been investigated previously. The degree of accuracy of a result can be measured in two ways. The first is by stating the relative error; this method is accurate and precise. The second is by stating the number of significant figures. This method is far less accurate. For instance, $10 \cdot 1$ and $99 \cdot 9$ are both three significant figure results, but the second is 10 times as accurate as the first. As, however, the method of significant figures is the one generally used, I shall base my investigation on that. My investigation will consider the degree of error produced in an answer by using numbers which themselves have an error.

There is another kind of error which is caused by forcing a result to a given number of significant figures. This is an entirely random effect and is produced even if the unforced number is completely free of error. We are not concerned with this artificially produced error. What we are concerned with is the error in an answer which is the natural consequence of errors in the original numbers.

It is clear that the degree of error does not depend on the position of the decimal point. I shall therefore find it convenient, in a multiplication or division, to express the original numbers in standard form—that is to say, express them as numbers between 1 and 10.

Multiplication. Each number correct to n figures.

I start with the multiplication of two numbers, a and b, each in standard form and each correct to n significant figures. Let e_1 and e_2 be the errors in the nth figures so that e_1 and e_2 are each between $\pm \frac{1}{2}$. If E_1 and E_2 are the actual errors in a and b, $E_1 = e_1 \times 10^{-n+1}$ and $E_2 = e_2 \times 10^{-n+1}$. The error in the product ab is therefore $(a + E_1)(b + E_2) - ab = aE_2 + bE_1$, neglecting the term $E_1 E_2$. The error in the $(n-1)$th decimal place is $ae_2 + be_1$ and in the nth figure as follows:

Case	Error in nth figure	Max. error in the nth figure
$ab < 10$	$ae_2 + be_1$	$5\frac{1}{2}$ (when $a = 10$, $b = 1$)
$ab > 10$	$(ae_2 + be_1)/10$	1 (when $a = b = 10$)

The facts, so far, are well known but previous investigations have not gone beyond this point. The usual argument has been to take the figures in the last column as a measure of the unreliability of the answer, in other words, the maximum error has been regarded as the kind of error that is usual and to be expected; this is the view which had been accepted practically universally. Now we have seen that, in adding several numbers, the maximum error is not the error that occurs in practice and that, in fact, there is not the remotest chance of getting anywhere near it. It must be clear that in multiplication, also, the maximum error is not a measure of the unreliability of the answer. It is clear that the only satisfactory way is to find the probability that any given error will be exceeded. Probability is, of course, used in the solution of many mathematical problems. It is the method of stating the solution of a wide variety of problems in statistics; the results are reliable and consistent and mathematicians accept them with confidence. Likewise, we may accept, with perfect confidence, a solution of the problem we are considering in terms of probability. If an error is unlikely we must be prepared to accept a result on the assumption that it will not occur.

What we want to know, then, is the probability of an assigned error being exceeded. Among the questions we should like answered are : (i) which error is as likely not to be exceeded as exceeded? (ii) which error is it reasonably safe to assume will not be exceeded? *e.g.* a chance of less than 1 in 20, which is one usually accepted in statistics.

I started on this subject by trying to solve the particular problem of the probability that the error in the nth figure exceeded 5. For the present I shall adopt the procedure of stating the results of my investigation without giving proofs. I shall defer all questions of proofs till later. The probability, then, that the error in the nth figure exceeds 5 is 1/143,000 (see Appendix 4). To make sure that such an error is exceeded it would be necessary to do a multiplication every day for 400 years. The absurdity of going by the maximum error is therefore obvious. I worked out one or two other cases, also as particular problems. The results of these are given in the following table :

Error in nth figure exceeds	5	$3\frac{1}{2}$	1	0
Probability - - -	1/143000	1/530	1/12·4	1/1

The calculations involved in obtaining these values were considerable. I therefore tried the possibility of obtaining other values, even if somewhat rough, by graphical interpolation. It is sometimes possible to get a reasonable graph with three or four points. (We can plot four points here.) A scale which extended to 143,000 would crowd the other three points to nothingness. In fact no ordinary scale would be of any use, and graphical interpolation seems to be impossible. But is it? If it seems so, it is because we have confined our attention, too much, to ordinary graphical papers and have failed to realise the possibilities of logarithmic and semi-logarithmic papers. Let us see what the points give when plotted on semi-logarithmic paper. This is given in Fig. 2. We have with only four points (seemingly impossible points) a graph which is quite reasonable. It is like taking a photograph in the dark with infra-red photographic film ; the picture, previously invisible, suddenly springs into life. The reason for the kink in the graph is that, down to the value in the error scale of 1, the case of $ab > 10$ has made no contribution. If we had continued to ignore any contribution from this case, the graph would have come out on the N scale at 5·79, 1/5·79 being the probability that the case $ab < 10$ occurs at all. (This is easily proved, see Appendix 1.) It is remarkable that, with so few data, it is possible to interpolate over the wide range of 1 to 100,000 to within an accuracy of 2 or 3%. With a few more plotted values, an accuracy of 98 to 99% would have been possible throughout the whole range. This indicates the value of logarithmic scales for many purposes. I have an elementary book on radio which demands from the reader nothing higher than School Certificate mathematics. Out of the 66 graphs in the book, 26% use logarithmic scales. Does this not suggest that class-room mathematics is lagging behind mathematics as used in practice?

In this graph (Fig. 2) the cases $ab > 10$ and $ab < 10$ are lumped together. We can answer now the questions put earlier. (1) What error in the nth figure is as likely not to be reached as exceeded? The answer is only about 0·2. (2) For what error in the nth figure is the probability of it not being exceeded equal to 1/20? The answer is 1·5. It is thus seen that the results of a multiplication with approximate numbers is far more reliable than had hitherto been supposed. If we consider the cases $ab > 10$ and $ab < 10$ separately, the former case has an accuracy far greater than in the lumped results. The maximum error in the nth figure is 1 and the probability that it exceeds $\frac{1}{2}$

can be proved to be about 1/20. An answer to n figures can therefore always be considered as reliable. If we are given that $ab < 10$ the probability values should be increased in the ratio of $1/5\cdot8$. Even then the probability that the error in the nth figure exceeds 2 is $1/6\frac{1}{2}$, and that it exceeds $2\frac{1}{2}$ is $1/14$. This brings me to an important point. If it is known that the nth figure of a result has a probable error of 2, should the answer be given to n or $n-1$ figures? The case for the former view has been admirably argued by H. Berry in the *Mathematical Gazette* (May, 1933) and I am in complete agreement with it. I shall give but one illustration in support of this view. Suppose we have an answer $6\cdot324$ with a possible error of 2 in the last figure. We know precisely what is meant by the answer $6\cdot324 \pm 0\cdot002$; but the answer $6\cdot32$ means anything between $6\cdot315$ and $6\cdot325$ with the range of uncertainty 5 times as great. In any case, we cannot be sure that $6\cdot32$ is correct to even 3 significant figures as the 4-figure answer may actually be $6\cdot326$. The 4-figure answer is therefore in every way more satisfactory. In this connection, the habit, which has widely developed, of dropping the 4th figure in a logarithmic computation is to be strongly condemned. It is like throwing away a half-inch thick peeling of a vegetable together with valuable vitamins. We can therefore sum up by saying that in a multiplication of two numbers, each correct to n figures, the answer should be given to n figures.

Division. Each number correct to n figures.

Denote the division by a/b. As before, suppose a and b each correct to n figures, that the error in the nth figures are respectively e_1 and e_2 where e_1 and e_2 are each between $\pm\frac{1}{2}$ and that the actual errors are E_1 and E_2 so that $E_1 = e_1 \times 10^{n-1}$ and $E_2 = e_2 \times 10^{n-1}$. The maximum error in the quotient is $(a + E_1)/(b - E_2) - a/b$. This is equal to $(aE_2 + bE_1)/b^2$ so that the error in the $(n-1)$th decimal place is $(ae_2 + be_1)/b^2$. If $a < b$, the quotient is less than 1. The error in the nth figure is therefore as follows:

Case	Error in nth figure	Max. error in nth figure
$a < b$	$10(ae_2 + be_1)/b^2$	10 (when $a = b = 1$)
$a > b$	$(ae_2 + be_1)/b^2$	$5\frac{1}{2}$ (when $a = 10$, $b = 1$)

My first investigation was to find the probability that the error in the nth figure exceeded 5. This worked out as 1/6000 (see Appendix 3), a result which is hard to believe. This and the results of other calculations are given below:

Error in nth fig. exceeds -	5	2	1	0
Probability - - -	1/6000	1/27·9	1/8·02	1/1

These results were graphed as in the case of multiplication (see Fig. 2). The graph showed that there was just as much chance of the error in the nth figure being below $0\cdot2$ as above, the same result as in multiplication. The fact that the probability of the error in the nth figure exceeding 1 is only $1/8$ and of exceeding 2 is $1/28$ gives us the rule that the quotient should be given to n figures, as in multiplication.

One number correct to more than n figures.

If one of the numbers is accurate to more than n figures it would be a reasonable assumption to ignore the error in the more accurate number. This has the advantage that the calculations involved are considerably simplified. The next table gives the results of the calculations. For sim-

plicity I have lumped together the different cases of multiplication and I did the same for division.

Error in nth fig. exceeds - -	5	4	3	2	1	$\frac{1}{2}$	0·15
Proby. Mult. (lumped) - -	1/∞	1/4860	1/470	1/98	1/24$\frac{1}{2}$	1/11	1/2
Prob. Div. (lumped) - -	1/∞	1/13900	1/1190	1/202	1/32	1/7·3	1/2·1

As would be expected, these results indicate greater reliability than when both numbers are correct to n figures. We should have no hesitation in giving the answer to n significant figures and the degree of reliability in the answer is fairly high.

Proofs.

I propose, now, to give some indication of the method of analysis used in arriving at the results. Owing to limitations of space I can give only a few details. The easiest case is when one of the two numbers can be considered free of error, and I shall take the case of multiplication. As an illustration we shall investigate the probability that the error in the nth figure exceeds 2.

Case $ab < 10$. The error in the nth figure is $ae_2 + be_1$, but $e_1 = 0$. Thus $ae_2 > 2$, whence $e_2 > 2/a$ and $a > 2/e_2$. Hence $a > 4$, so that a varies from 4 to 10. Thus the variations of a, b, and e_2 are :

$$a \text{ between 4 and 10,}$$
$$b \text{ between 1 and } 10/a,$$
$$e_2 \text{ between } 2/a \text{ and } \tfrac{1}{2}.$$

The probability that a is in the region δa is $\delta a/9$, that b is in the region δb is $\delta b/9$, that $|e_2|$ is in the region δe_2 is $\delta e_2/\tfrac{1}{2}$. Therefore the probability that a, b, and e_2 are simultaneously in the regions δa, δb, δe_2 is

$$\frac{\delta a}{9} \times \frac{\delta b}{9} \times \frac{\delta e_2}{\tfrac{1}{2}},$$

and the total probability is

$$\lim \Sigma \frac{\delta a}{9} \times \frac{\delta b}{9} \times \frac{\delta e_2}{\tfrac{1}{2}},$$

or

$$\tfrac{2}{81} \int_{a=4}^{10} \int_{b=1}^{10/a} \int_{e_2=2/a}^{\tfrac{1}{2}} da\, db\, de_2 = 0\cdot8281/81 = 0\cdot0103.$$

Case $ab > 10$. The error in the nth figure is $(ae_2 + be_1)/10 = ae_2/10$. Thus $ae_2/10 > 2$, so that $ae_2 > 20$ or $a > 40$. Hence this error cannot occur if $ab > 10$ and the total probability for both cases is 0·0103.

Errors in both numbers.

When both numbers are correct to n significant figures, the analysis is intricate and the calculations are very lengthy. In the case of division, $a < b$, if we wish to calculate the probability that the error in the nth figure exceeds 1, we must evaluate the integral

$$\tfrac{1}{81} \iiiint da\, db\, de_2\, de_1,$$

and four others like it, where in the given integral the limits of integration are

$$\text{for } e_1, \text{ from } b/10 - ae_2/b \text{ to } \tfrac{1}{2};$$
$$\text{for } e_2, \text{ from } (b^2 - 5b)/10a \text{ to } \tfrac{1}{2};$$
$$\text{for } b, \text{ from } a \text{ to } \tfrac{1}{2}\{5 - \sqrt{(25 - 20a)}\};$$
$$\text{for } a, \text{ from } \tfrac{5}{4} \text{ to } 8.$$

If the error in the nth figure exceeds $0{\cdot}6$ it is necessary to evaluate seven such integrals. For some errors in multiplication as many as thirteen integrals must be evaluated. On the other hand, for other errors, the evaluation of one integral is sufficient. It is quite impossible in the space at my disposal to go into the analysis, but I shall give the analysis of the division case where the error exceeds 5 in the nth figure and the multiplication case of the same error, in each of which the evaluation of one integral is sufficient (see Appendix 3 and 4). As against this, I shall give most of the analysis for the remainder of this paper.

Number of processes unlimited.

The rest of my paper deals with combined multiplication and division without restriction as to the number of processes involved. It is quite clear that, owing to the complexity of the analysis, the extension of the method, so far used, to even two processes, is quite unthinkable. There is another possible method of approach, namely by a consideration of the relative error. This method, it was noted, achieves a precision which is not reached by the method of significant figures. There is also a great gain in simplicity. The method of significant figures involves a consideration of four cases, two in multiplication (*e.g.* $ab > 10$ and $ab < 10$) and two in division, whereas in the method of relative error we need consider only one case irrespective of whether we multiply or divide. If the relative error of two numbers are respectively r_1 and r_2 the relative error of the answer is always $r_1 + r_2$. If we compute with three numbers, with relative errors r_1, r_2 and r_3 the relative error of the answer is always $r_1 + r_2 + r_3$ irrespective of whether the processes are mixed or not ; similarly for any number of processes. The maximum relative error of one number is $5/10^n$. Hence in a computation with two numbers the maximum relative error is $5/10^n \times 2$, with three numbers it is $5/10^n \times 3$, and so on. Starting with the case of two numbers, and using methods similar to those already referred to, I found that the probability that the relative error exceeded $5/10^n$ was $1/476$. In nearly all cases, however, the calculations required the evaluation of eight integrals. I was therefore obliged to approach the problem from another angle. In this I constructed a function $P_1(x)$ which I shall call a probability function. This function refers to errors of a single number : it gives the probability that the relative error in a single number exceeds $x/10^n$. The function is :

$$(5 - x)^2/90x, \quad \text{when } 5 > x > \tfrac{1}{2},$$
and
$$1 - 1{\cdot}1x, \quad \text{when } \tfrac{1}{2} > x > 0. \quad \text{(See Appendix 2.)}$$

The figure opposite gives the graph of the function, $P_1(x_1)$ for the number a, and an equivalent graph of the function $P_1(x_2)$ for the number b can be drawn. By means of the two functions $P_1(x_1)$ and $P_1(x_2)$, or their graphs, we can work out the probability that the relative error in a computation with two numbers exceeds a given value. Thus suppose we wish to calculate the probability that the relative error exceeds $5/10^n$. This means that if $x_1/10^n$, $x_2/10^n$ are the relative errors of a and b, $x_1 + x_2 > 5$. If the value of x_1 is confined to the region δx_1, then the value of x_2 must be greater than $5 - x_1$. The probability that x_1 is in the region δx_1 is $|\delta P_1(x_1)|$ and the probability

that x_2 exceeds $5 - x_1$ can be calculated from $P_1(x_2)$ by giving x_2 the value $(5 - x_1)$. This is $[5 - (5 - x_1)]^2/90(5 - x_1) = x_1^2/90(5 - x)$. Also, $|\delta P_1(x_1)|$ is $(25 - x_1^2)\delta x_1/90x_1^2$ so that $|\delta P_1(x_1)| \times P_1(5 - x_1) = (5 - x_1)\delta x_1/8100$. Hence, the required probability is $\frac{1}{2}\left| \dfrac{(5 - x_1)dx_1}{8100} \right.$. The factor $\frac{1}{2}$ is added because, for the particular error we are considering, the values of x_1, x_2 must have the same sign. Suitable allowances must be made when x_1 falls within the range 0 to $\frac{1}{2}$ and when x_2 falls within that range, because the form of the function changes and there is a corresponding change in the form of the integral. On completing the evaluation, the answer obtained was $1/476$. This was exactly
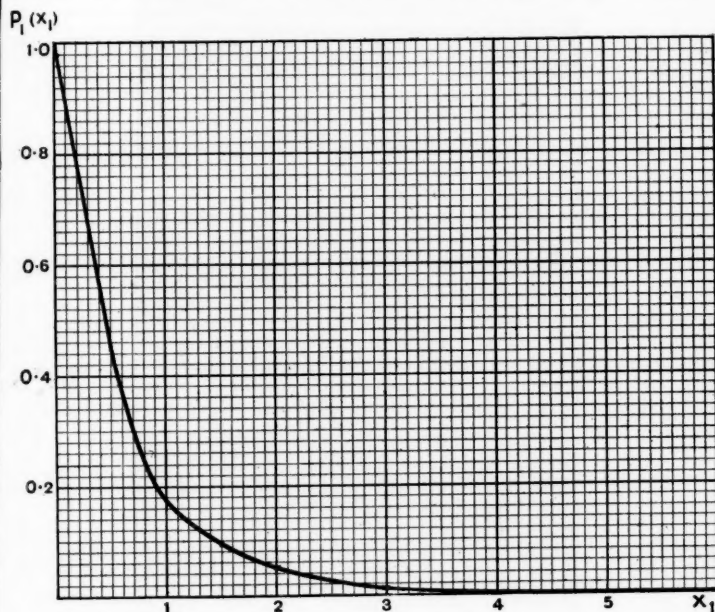
$P_1(x_1)$

Fig. 1

Graph of $P_1(x_1)$.

the same as was obtained by a totally different method. This, then, was not only a check on the calculations, but a verification of the soundness of both the theories on which the calculations were based. The answer was also obtained approximately by dividing the x_1 scale into 20 equal parts. $|\delta P_1(x_1)|$ was obtained as the difference of consecutive ordinates, and $P_1(x_2)$ or $P_1(5 - x_1)$ was obtained by giving x_1 the mid-value of the corresponding interval on the x_1 scale. The summation gave the satisfactory value of $1/475$. This method has important consequences as will be seen later. The approximate method was used to calculate the probabilities that the relative error will exceed $1/10^n$, $2/10^n$, ... $9/10^n$ and the values obtained were used to draw the graph of what I shall call the Second Probability Function, $P_2(x)$ (Fig. 5). By
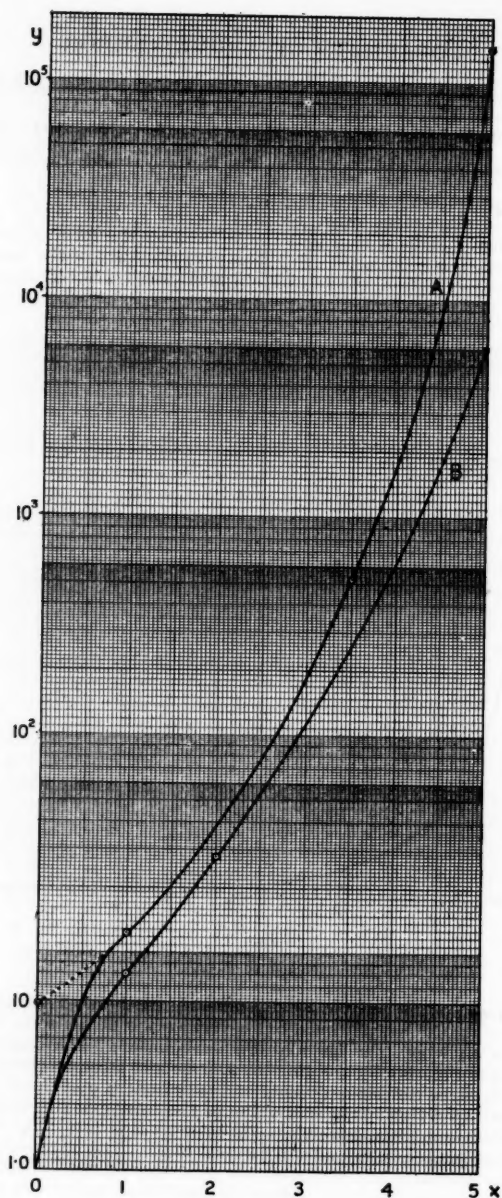
Fig. 2.

Multiplication and division of two numbers each correct to n figures.
The probability that the error in the nth figure of the answer exceeds x is $1/y$
Graph A : multiplication.   Graph B : division.

means of this graph it is possible to read off the probability that the relative error in a computation with two numbers will exceed any assigned value.

We are now in a position to proceed as far as we like. If we have four numbers a, b, c, d, with relative errors $r_1/10^n$, $r_2/10^n$, $r_3/10^n$, and $r_4/10^n$, the relative error of the answer is $(r_1+r_2+r_3+r_4)/10^n$. Let $x_1=r_1+r_2$ and $x_2=r_3+r_4$. From the second probability function, $P_2(x_1)$, we can obtain the probability that $r_1+r_2>x_1$ and from the graph of an equal second probability function, $P_2(x_2)$, we can obtain the probability that $r_3+r_4>x_2$. A little while back we made use of two first probability functions, $P_1(x_1)$ and $P_1(x_2)$, to obtain the probability that x_1+x_2 was greater than any assigned value, say 5. In like manner by dividing the x_1 scale into 20 equal parts we can combine two second probability graphs, $P_2(x_1)$ and $P_2(x_2)$, to obtain the probability that x_1+x_2 is greater than any assigned value. But $x_1+x_2=r_1+r_2+r_3+r_4$. Hence we can obtain the probability that $r_1+r_2+r_3+r_4$ is greater than any assigned value. Using this method I calculated the probabilities that the relative errors in a computation with four numbers were greater than $1/10^n$, $2/10^n$, $3/10^n$, and so on. With these values I drew the graph of the fourth probability function. This enabled one to read off the probability that the relative error of an answer in a computation with four numbers exceeded any assigned value. I likewise combined a second probability function with a first probability function to obtain the third probability function for three number computations and I combined two fourth probability functions to obtain the eighth probability function for computations with eight numbers. There is no limit to this process. Fig. 3 is a graph of the fourth probability function. The maximum relative error is $20/10^n$. From the graph it is seen that the probability that the relative error will exceed only half this value is $1/56000$. The probability that it will exceed a quarter of the maximum is $1/80$, and the probability that it will exceed $1/10$th of the maximum is $\frac{1}{4}$. There is an even chance that it will not exceed $1\cdot1/10^n$. With an eight-number computation, in which the maximum relative error is $40/10^n$, the probabilities that the relative error will not exceed $x/10^n$ for different values of x are as follows :

Values of x -	10	4	1·65	5
Probability -	1/18000	1/8·5	1/2	1/18

The maximum relative error with two numbers is $10/10^n$. The probability that it will exceed $9/10^n$ is about $1/2,000,000$.

The general probability function.

Consider any number, a, after its conversion to standard form. If the error in the nth figure is e, e being between $\pm\frac{1}{2}$, the relative error is $e/(a\times10^{n-1})$ or $10e/a10^n$. We need consider only the value of $10e/a$. This can have any value between 0 and 5. If we have a large number, M, of such errors, the standard deviation, σ, of these errors is $\sqrt{\left\{\frac{1}{M}\Sigma100e^2/a^2\right\}}$. As e can vary between 0 and $\frac{1}{2}$, the number of times δe is contained in the range 0 to $\frac{1}{2}$ is $\frac{1}{2}/\delta e$; and the number of times δa occurs within the range 1 to 10 is $9/\delta a$. Hence $M=2/\delta e\times9/\delta a$, and $\sigma=\sqrt{\left\{\frac{2}{9}\int_{a-1}^{10}\int_{e=0}^{\frac{1}{2}}\frac{100e^2}{a^2}\,da\,de\right\}}=0\cdot913$.

This is the standard deviation for one number. If we have N numbers we must have a factor N under the square root sign, and the standard deviation becomes $0\cdot913\sqrt{N}$. As N increases, the frequency function of relative errors approaches more and more that of the curve of normal error. Hence, if we
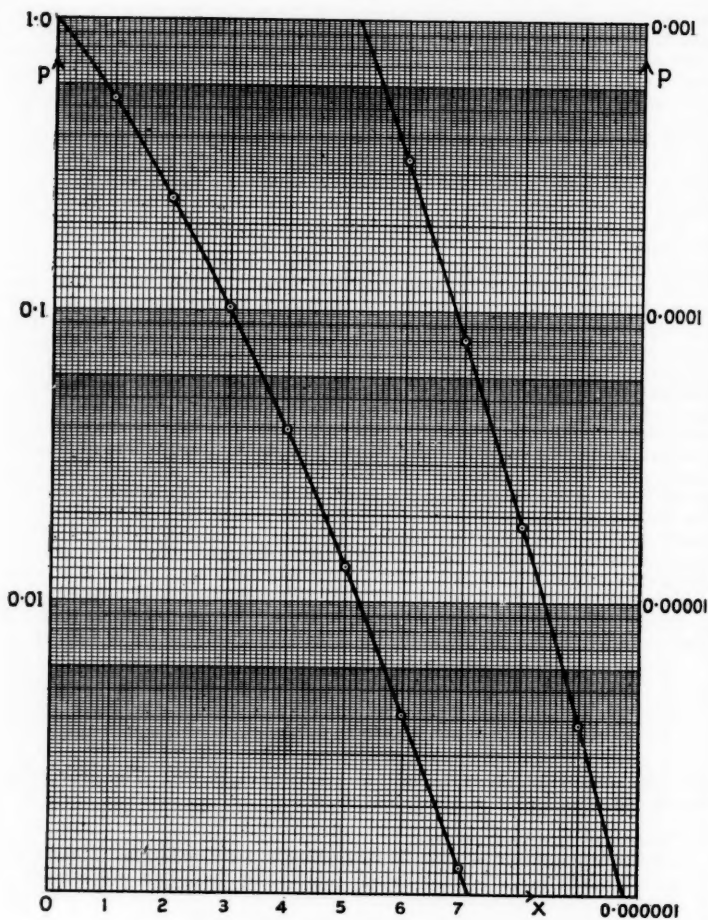
Fig. 3.

Fourth probability function.
P = probability that the relative error exceeds $x/10^n$.
The graph on the right is a continuation of that on the left.

have a computation with N numbers, the probability that the relative error exceeds $k\sqrt{N}/10^n$ approaches, as N increases, the value

$$1 - \frac{2}{\sigma\sqrt{(2\pi)}} \int_0^k e^{-x^2/2\sigma^2} dx.$$

This function is therefore the Generalised Probability Function. It is an approximation to all the other Probability Functions we discussed previously. The approximation becomes increasingly better as N increases. It is seen that the graph of even the First Probability Function is not unlike the positive half of the curve of normal error. We should therefore expect that the Generalised Probability Function becomes a good approximation to the other Probability Functions for even moderately high values of N. The highest Probability Function I calculated was the Eighth. How close an approximation to this is the Generalised Probability Function? Fig. 4 gives a comparison of the graph of the Eighth Probability Function with that obtained from the Generalised Probability Function. It is seen that for quite a long stretch it is hardly possible to distinguish between the two graphs. The value of x for the first graph cannot exceed 40 while that of the second graph goes on to infinity, so that the graphs cross again before finally separating. In view of the close agreement between the two graphs when N is 8, it is worth examining what closeness there is for lower values of N. The next diagram (Fig. 5) gives comparisons for $N = 4$ and $N = 2$ (the lowest value possible). There is good agreement for quite a useful range for $N = 4$ and rough agreement for $N = 2$. The Generalised Probability Function can therefore serve as a general test of the reliability of an answer in a computation consisting of multiplication and division. This says that if the number of numbers in a computation goes up N-fold, the relative error goes up $\sqrt{N}$-fold, which brings us back full circle to the rule applicable to addition. It also says that the relative error in a computation of 7 operations is only twice as great as that with one operation ($N = 8$ for 7 operations and 2 for one operation); moreover, it is seldom that a computation involves more than 8 numbers. But the error is smaller even than a twofold increase as the following table, based on the second and eighth probability functions will show. This table gives the probabilities that the relative errors will exceed $x/10^n$ for different values of x when $N = 2$ compared with the corresponding probabilities when $N = 8$ and x is doubled.

x	2	3	4	5	$5\frac{1}{2}$
Probability $N = 2$ -	0·114	0·384	0·0107	0·00210	0·00091

x	4	6	8	10	11
Probability $N = 8$ -	0·117	0·249	0·00415	0·00056	0·00019

To put it another way, for what values of x are the probablities 0·1, 0·01 and 0·001 for the cases $N = 2$ and $N = 8$? The following table shows this.

Probability -	0·1	0·01	0·001
x when $N = 2$ -	2·2	4·04	5·44
x when $N = 8$ -	4·2	7·04	9·40

Thus, although the errors in a single process are much less than have hitherto been supposed, we are very unlikely to encounter errors which are twice as
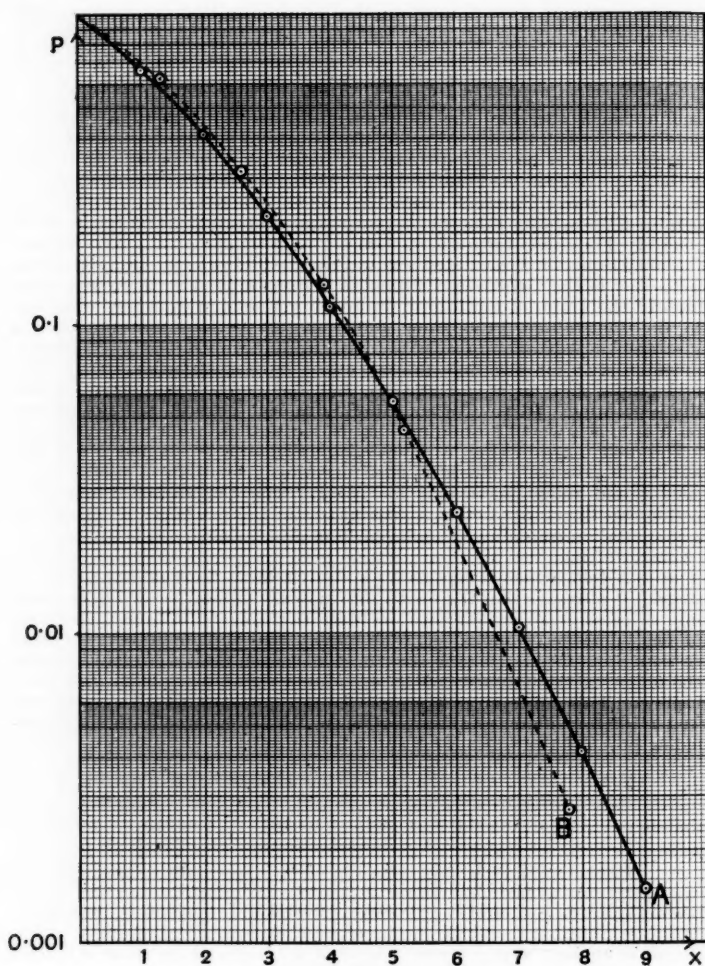
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FIG. 4.

Comparison between the 8th probability function and the
generalised probability function for $N = 8$.

Graph A : 8th probability function.   Graph B : generalised probability function.

great as these, and with a three-number operation they are only $1 \cdot 2$ times as great.

In mathematics the "natural" unit of measurement is bound to emerge eventually. In angle it is the radian; in logarithms and exponentials it is the number e. This investigation is a clear indication that the "natural" method of measuring the degree of error is the method of relative error and not the error in the nth figure. An error of 2 in 101 is far more serious than in 984. For further remarks on this subject I would refer the reader to an article of mine in the *Mathematical Gazette* of December 1932. This article, written from the elementary standpoint, is complementary to the present article.

I cannot close without a cautionary remark. Be on your guard when squaring. In the ordinary way plus and minus errors tend to cancel each other. In squaring, however, there is a bias and the relative error is doubled. If this process is continued (an occurrence which is rare), the relative error increases 4-fold, 8-fold and so on. On the other hand, if the operation is a square root, the relative error is halved. If this process is continued, the relative error becomes $\frac{1}{4}$, $\frac{1}{8}$, and so on, of the original relative error. Thus, we might start with a number correct to three significant figures and, in the course of time, the answer will be correct to 10 significant figures!

APPENDIX

1. *Probability that $ab < 10$* (see pp. 101).

$$\text{Probability} = \frac{1}{81} \int_{a-1}^{10} \int_{b-1}^{10/a} da\ db = \frac{14 \cdot 02}{81} = 1/5 \cdot 79.$$

2. *First Probability Function. To find the probability that the relative error of a number exceeds $x/10^n$ if the number is correct to n significant figures.*

If the number is a and the error in the nth figure is e, the relative error is $e/a \cdot 10^{n-1}$ and thus

$$e/a > x/10. \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\dotfill(1)$$

Thus $a < 5/x$ and hence

$$\left.\begin{array}{l} \text{if } \tfrac{1}{2} < x < 5,\ a \text{ varies between 1 and } 5/x \\ \text{if } x < \tfrac{1}{2}, \quad a \text{ varies between 1 and 10} \end{array}\right\} \quad \dotfill(2)$$

From (1), $e > ax/10$ and so $|e|$ varies from $ax/10$ to $\frac{1}{2}$. The probability that $|\delta e|$ is in the range 0 to $\frac{1}{2}$ is $\delta e/\frac{1}{2}$, and the probability that δa is in the range 1 to 10 is $\delta a/9$. Hence if $\frac{1}{2} < x < 5$, the total probability is

$$\int_{a-1}^{5/x} \int_{e-ax/10}^{\frac{1}{2}} \tfrac{2}{9} da\ de = (x-5)^2/90x,$$

and if $0 < x < \frac{1}{2}$, the total probability is

$$\int_{a-1}^{10} \int_{e-ax/10}^{\frac{1}{2}} \tfrac{2}{9} da\ de = 1 \cdot 1x.$$

3. *Division of a by b. Probability that the error in the nth figure exceeds 5.*

Case $a < b$. The error in the nth figure is $10(ae_2 + be_1)/b^2$.

Hence
$$ae_2 + be_1 > \tfrac{1}{2}b^2. \quad\quad\quad\quad\quad\quad\quad\dotfill(1)$$

Thus $a + b > b^2$, or $b^2 - b - a < 0$.

Hence ab is between the roots $\frac{1}{2}\{1 \pm \sqrt{(1 + 4a)}\}$ or, as the negative root is inadmissible,

$$b < \tfrac{1}{2}\{1 + \sqrt{(1 + 4a)}\}.$$

But $b > a$ and so

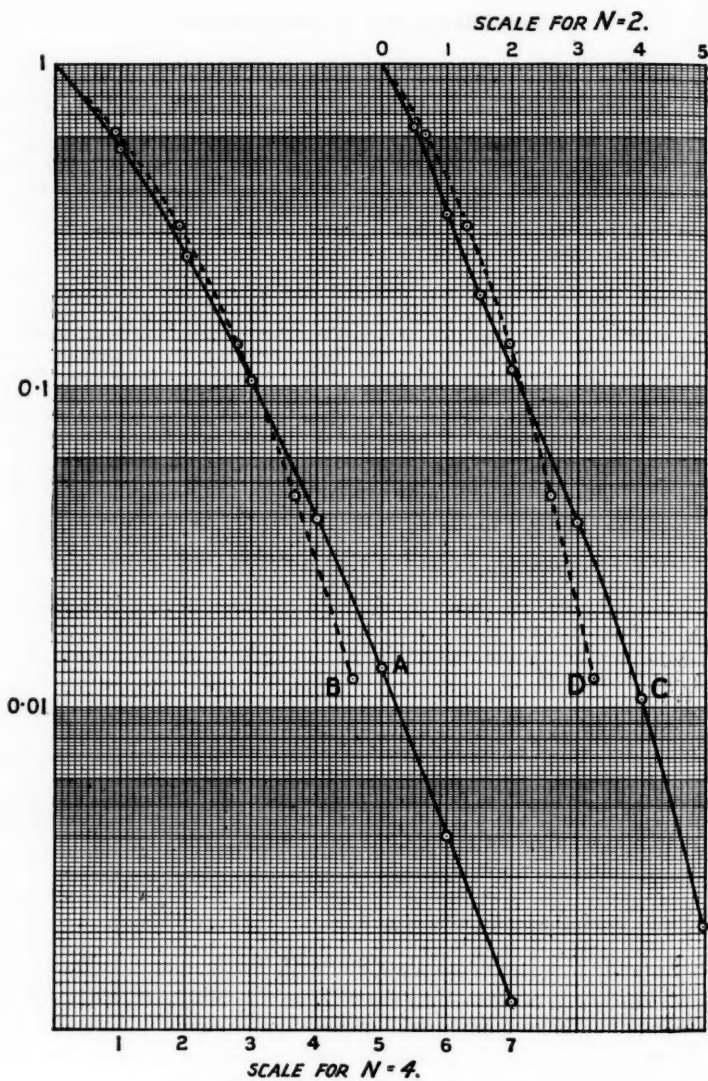$$a < \tfrac{1}{2}\{1 + \sqrt{(1 + 4a)}\},$$

Fig. 5.

A is graph of fourth probability function,
B is graph of general probability function,
C is graph of second probability function,
D is graph of general probability function.

5

which on simplification becomes

$$a < 2.$$

Values of e_1.   From (1) $e_1 > \tfrac{1}{2}b - ae_2/b$,

or $e_1 > \tfrac{1}{2}b - a/2b$.

Values of e_2.   From (1) $e_2 > b^2/2a - be_1/a$.

Hence,

e_2 varies between $(b^2/2a - be_1/a)$ and $\tfrac{1}{2}$,

e_1 varies between $(\tfrac{1}{2}b - a/2b)$ and $\tfrac{1}{2}$,

b varies between a and $\tfrac{1}{2}\{1 + \sqrt{(1 + 4a)}\}$,

a varies between 1 and 2.

The probabilities that a, b, e_1, e_2 are within the ranges δa, δb, δe_1, δe_2 are respectively $\delta a/9$, $\delta b/9$, $\delta e_1/1$, $\delta e_2/1$, and the total probability is

$$\tfrac{2}{81}\iiiint da\, db\, de_1\, de_2,$$

where the range of each variable is that given above. The quadruple integral is then approximately equal to $1/6000$. The factor 2 is to allow for the case when the errors are both negative. Some of the finer points of the argument are omitted, although this would lead to serious errors in investigating the probabilities of other errors.

Case a/b in which $a > b$. This can be investigated similarly. The probability in this case works out to the value $1/343,000$, which scarcely affects the answer for both cases taken together.

4. *Multiplication.* $ab < 10$. *The probabilities that the error in the nth figure exceeds* 5.

We have $ae_2 + be_1 > 5.$ ....(1)

Thus $a + b > 10$. As $b < 10/a$, we have $a + 10/a > 10$, or $a^2 - 10a + 10 > 0$.
Thus a cannot have values between $5 \pm \sqrt{15}$.
If $a < 5 - \sqrt{15}$, $b > 10 - a$ gives $b > 5 + \sqrt{15}$.
If $a > 5 + \sqrt{15}$, $b < 10/a$ gives $b < 5 - \sqrt{15}$.
The first condition corresponds to $a < b$, the second to $a > b$. It is sufficient to take the former only and double the probability as a is as likely to be less than b as greater.

Values of e_2.   From (1), $e_2 > (5 - be_1)/a$,

or $e_2 > 5/a - b/2a$.

Values of e_1.   From (1), $e_1 > (5 - ae_2)/b$.

Hence the variations of a, b, e_1, e_2, are

a varies between 1 and $5 - \sqrt{15}$,

b varies between $10 - a$ and $10/a$,

e_2 varies between $(10 - b)/2a$ and $\tfrac{1}{2}$,

e_1 varies between $(5 - ae_2)/b$ and $\tfrac{1}{2}$.

The probability must be doubled to allow for negative errors, and again doubled to allow for $b < a$. Hence the total probability is given by the quadruple integral taken over the range for each variable just given:

$$\tfrac{4}{81}\iiiint da\, db\, de_2\, de_1 = \frac{1}{143,000}.$$

S. I.

H

MATHEMATICAL NOTES.

2131. *The three classes of lever and a family of mnemonics.*

1. The terms Fulcrum, Power, and Weight are (mentally) arranged in alphabetical order; a column of three F's is written (a); then a column of three P's; three W's are then written, not in a column, but starting at the top left of the letters already written and moving to the right as they descend.

(a)	F		(b)	F	P		(c)	W	F	P
	F			F	P				F	W	P
	F			F	P				F		P	W

The nth row of this pattern (c) then shows the arrangement of elements in a lever of the nth class.

2. The same mnemonic is available in French or German as the alphabetical order F, P, W is preserved in translation!

Fulcrum	point d'Appui	Drehpunkt
Power	Puissance	Kraft
Weight	Resistance	Last

This may not occur with other foreign languages; again, the mnemonic is useless in English if one is accustomed to use terms other than Fulcrum, Power, and Weight. A variety of similar devices have therefore been constructed; in all cases the pattern finally written will display in its nth row a lever of the nth class.

3. In German the word " Stuetze " is sometimes used for " fulcrum ". To meet this case write a central column in alphabetical order from the bottom upwards (a); complete all rows in either possible way (b) to obtain the desired pattern:

(a)	S (tuetze)	(b)	L	S	K	or	K	S	L
	L (ast)		S	L	K		etc., etc.
	K (raft)		S	K	L		

4. The terms Effort, Fulcrum, and Resistance are used in some American textbooks; the following is appropriate for these terms: Write a row in alphabetical order (a); copy the first two letters twice each to form two columns (b); complete the pattern by moving the last letter to the left as it descends (c).

(a)	E	F	R	(b)	E	F	R	(c)	E		F	R
					E	F				E	R	F
					E	F			R	E		F

5. In Italian, the conventional words are:

Fulcrum	punto d'Appoggio	Fulcrum
Weight	Peso	Load
Power	Potenzo	Power

the alphabetical order agreeing with that of the English synonyms in the third column. In this case write a central column of three letters in alphabetical order (from top to bottom) and complete any row in either of the ways possible with the remaining letters.

L	F	P		Ap	
P	L	F	or	Pe	etc.
F	P	L		Po	

6. A further possibility is illustrated by the terms Support, Thrust, and Resistance. For these write a row in order (a); copy to form the last two

columns (b); complete the pattern by advancing the first letter to the right as it descends.

(a) R   S   T (b) R   S   T (c) R   S   T
                             S   T   S   R   T
                             S   T   S   T   R

7. The remaining case (*e.g.* Support, Power, Load) may well be left to readers interested in mnemonics and permutations.

8. The choice of conventional terms might well be governed by considerations of ease in learning the classes of lever and the simplest way to memorise this arbitrary classification is to use the terms in column 3 of the table in para. 5. The central element determines the class of lever, and the alphabetical order F, L, P is the order 1, 2, 3. " Weight " is in some respects not a good word to use in the study of levers, and more than a mnemonic would be gained if it were replaced by " Load ".

9. If students need to classify levers as 1st, 2nd or 3rd class, and if " weight " persists as the usual word, it may still be useful to remember " Load " as a synonym. The weaker student will no doubt be able to invent other mnemonics such as F, M, P with " Mass " as a synonym for " Weight " !

<div align="right">H. M. F.</div>

2132. *The prime line.*

In a euclidean plane let ξ, η, $\mathbf{u}$, etc., be vectors taken as the homogeneous coordinates of points or lines in a cartesian rectangular coordinate system.

1. If ξ and η are any two points on a fixed line $\mathbf{u}$, and ξ', η' their polars with respect to a certain conic, and if lines are drawn through ξ and η perpendicular to η' and ξ' respectively, then the locus of the point of intersection of these perpendiculars is a line $\mathbf{u}'$ and $\mathbf{u}$, $\mathbf{u}'$ are connected by a projective transformation.

Let the given conic have the equation $(\mathbf{xx}) \equiv \Sigma a_{ij} x_i x_j = 0$ $(a_{ij} = a_{ji})$,

then
$$\xi_i' = \Sigma d_{ij} \xi_j ;$$

if we denote the line through η perpendicular to ξ' by $\mathbf{v}$, then

$$v_1 = -(a_{21}\xi_1\eta_3 + a_{22}\xi_2\eta_3 + a_{23}\xi_3\eta_3)$$
$$v_2 = (a_{11}\xi_1\eta_3 + a_{12}\xi_2\eta_3 + a_{13}\xi_3\eta_3),$$
$$v_3 = (a_{21}\xi_1\eta_1 + a_{22}\xi_2\eta_1 + a_{23}\xi_3\eta_1) - (a_{11}\xi_1\eta_2 + a_{12}\xi_2\eta_2 + a_{13}\xi_3\eta_2).$$

If we interchange the letters ξ and η in the above, we obtain the coordinates of the line $\mathbf{v}'$ passing through ξ and perpendicular to η'.

Now since $\mathbf{u} = \xi \wedge \eta$, there is a non-zero scalar ρ such that

$$\rho\mathbf{u}' = \mathbf{v} - \mathbf{v}' = \{-a_{22}u_1 + a_{12}u_2, \quad a_{21}u_1 - a_{11}u_2, \quad a_{31}u_1 + a_{32}u_2 - (a_{11} + a_{22})u_3\},$$

which says that the locus of the point of intersection of the lines $\mathbf{v}$ and $\mathbf{v}'$ is a line $\mathbf{u}'$, where $\mathbf{u}$ and $\mathbf{u}'$ are connected by the relations

$$\left. \begin{aligned} \rho u_1' &= -a_{22}u_1 + a_{12}u_2, \\ \rho u_2' &= a_{21}u_1 - a_{11}u_2, \\ \rho u_3' &= a_{31}u_1 + a_{32}u_2 - (a_{11} + a_{22})u_3. \end{aligned} \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(i)$$

In general, this transformation is non-singular; then $\mathbf{u}$ and $\mathbf{u}'$ are in one-to-one correspondence, and we call $\mathbf{u}'$ the prime line or simply the " prime " of $\mathbf{u}$ with respect to the conic of reference $(\mathbf{xx}) = 0$.

The condition for the transformation (i) to be singular is

$$D \equiv \begin{vmatrix} -a_{22} & a_{12} & 0 \\ a_{21} & -a_{11} & 0 \\ a_{31} & a_{32} & -(a_{11} + a_{22}) \end{vmatrix} \equiv (a_{11} + a_{22})(a_{12}^2 - a_{11}a_{22}) = 0;$$

thus $D=0$ if the conic of reference is a rectangular hyperbola or a parabola (in either case degenerate or non-degenerate).

Consider first the case $D \neq 0$. It is evident that :

2. The primes of the lines of a pencil form a pencil and the two pencils are projective.

3. The primes of lines enveloping an algebraic curve of class n envelop another, in general.

4. The primes of a line with respect to all conics of a family having the same asymptotes are the same.

For : the transformation (i) is independent of a_{33}.

5. A necessary and sufficient condition that a line should be perpendicular to its prime is that it should be parallel to an asymptote of the conic of reference.

From (i) we have

$$\rho(u_1 u_1' + u_2 u_2') = -(a_{22}u_1{}^2 - 2a_{12}u_1 u_2 + a_{11}u_2{}^2),$$

and so $u_1 u_1' + u_2 u_2' = 0$ if and only if $\mathbf{u}$ passes through either of the points at infinity on the lines $a_{11}x_1{}^2 + 2a_{12}x_1 x_2 + a_{22}x_2{}^2 = 0$.

6. A line and its prime are conjugate with respect to the conic of reference if and only if the pole of the line lies on the director circle of the conic of reference, and hence its prime is tangent to the circle.

If we take the conic of reference as

$$a_{11}x_1{}^2 + a_{22}x_2{}^2 + a_{33}x_3{}^2 = 0, \quad \dots\dots\dots\dots\dots\dots\dots\text{(ii)}$$

then (i) reduces to

$$\rho u_1' = a_{22}u_1, \quad \rho u_2' = a_{11}u_2, \quad \rho u_3' = (a_{11} + a_{22})u_3, \quad \dots\dots\dots\text{(iii)}$$

The polar of a point $\mathbf{x}$ with respect to (ii) is given by

$$\mathbf{u} = \{a_{11}x_1, \quad a_{22}x_2, \quad a_{33}x_3\},$$

and it follows that

$$\mathbf{u}' = \{a_{11}a_{22}x_1, \quad a_{11}a_{22}x_2, \quad (a_{11} + a_{22})a_{33}x_3\} ;$$

then $\mathbf{u}$ and $\mathbf{u}'$ are conjugate if and only if $\mathbf{u}' \cdot \mathbf{x} = 0$, that is,

$$x_1{}^2 + x_2{}^2 + a_{33}\left(\frac{1}{a_{11}} + \frac{1}{a_{22}}\right)x_3{}^2 = 0, \quad \dots\dots\dots\dots\dots\dots\text{(iv)}$$

which is the equation of the director circle ; its line equation,

$$a_{33}(a_{11} + a_{22})(u_1{}^2 + u_2{}^2) + a_{11}a_{22}u_3{}^2 = 0,$$

can be found by eliminating $\mathbf{x}$ from $\mathbf{u}'$ and (iv). Hence $\mathbf{u}'$ is a tangent to (iv).

7. A necessary and sufficient condition for a line to be parallel to its prime is that it should be parallel to an axis of the conic of reference. If the conic is a circle, then any line is parallel to its prime, and conversely ; the distance of the line from the centre of the circle is half the distance of its prime from this point.

Take the equation of the conic as (ii) ; then, by (iii), $\mathbf{u}$ is parallel to $\mathbf{u}'$ if and only if $a_{11}u_1 u_2 = a_{22}u_1 u_2$, that is, if $u_1 = 0$, $u_2 = 0$ or $a_{11} = a_{22}$. Now when $a_{11} = a_{22}$, (iii) can be written $\gamma u_1' = u_1$, $\gamma u_2' = u_2$, $\gamma u_3' = 2u_3$, and evidently the distance of $\mathbf{u}'$ from the centre of the circle is twice that of $\mathbf{u}$, and they are on the same side of the centre.

8. A necessary and sufficient condition that the point of intersection of a line and its prime should lie on a fixed line is that the line should be tangent to a parabola with the two axes of the (non-circular) conic of reference as tangents ; and so also its prime.

Take the conic as (ii) ; since the point of intersection of $\mathbf{u}$ and $\mathbf{u}'$ is $\boldsymbol{\xi} = \mathbf{u}_{\wedge}\mathbf{u}'$, we have

$$\gamma_1\xi = a_{22}u_2u_3, \quad \gamma_2\xi = -a_{11}u_3u_1, \quad \gamma_3\xi = (a_{11}-a_{22})u_1u_2.$$

Then $\boldsymbol{\xi}$ lies on a fixed line $\mathbf{c}$ if and only if

$$c_1a_{22}u_2u_3 - c_2a_{11}u_3u_1 + c_3(a_{11}-a_{22})u_1u_2 = 0, \quad\ldots\ldots\ldots\ldots\ldots(v)$$

which is, in general, a parabola touching the lines $(1, 0, 0)(0, 1, 0)$.

Conversely, a parabola touching the axes of the conic of reference may be written as

$$\lambda_1u_2u_3 + \lambda_2u_3u_1 + \lambda_3u_1u_2 = 0 ;$$

comparing this with (v) we have $\mathbf{c}$ in terms of the a's and λ's and so $\mathbf{c}$ is a fixed line, which may be regarded as the locus of the point of intersection of $\mathbf{u}$ and $\mathbf{u}'$.

If we eliminate $\mathbf{u}$ from (iii) and (v) we obtain a form in $\mathbf{u}'$ similar to (v), and hence the last part of the result.

9. A line coincides with its prime if and only if it is an axis of the conic or the line at infinity.

In (iii), $\mathbf{u} = \mathbf{u}'$ if and only if $\mathbf{u}$ is one of the lines $(1, 0, 0)(0, 1, 0)(0, 0, 1)$, provided $a_{11}\neq a_{22}$. If the conic is a circle, any line through its centre is an axis.

10. The primes of a line with respect to a pencil of conics form a pencil of lines, in general ; but these primes all coincide if and only if the primes of the given line with respect to two conics of the pencil coincide.

This is easily proved.

Now consider the case $D \equiv (a_{11}+a_{22})(a_{12}{}^2 - a_{11}a_{22}) = 0.$

When $a_{11} + a_{22} = 0$, equation (ii) represents a rectangular hyperbola, degenerate $(a_{33}=0)$ or non-degenerate $(a_{33}\neq 0)$; (iii) reduces to

$$\rho u_1' = u_1, \quad \rho u_2' = u_2, \quad u_3' = 0.$$

Thus $\mathbf{u}'$ always passes through the centre of the hyperbola, for any $\mathbf{u}$; and $u_1'/u_2' = -u_1/u_2$, that is, $\tan\theta' = \tan(-\theta)$, where θ, θ' are the direction angles from the x_1-axis to $\mathbf{u}$ and $\mathbf{u}'$. Thus we have :

11. With respect to a rectangular hyperbola (degenerate or not), parallel lines have the same prime, and this passes through the centre of the hyperbola ; if the parallel lines make an angle θ with the transverse axis, the common prime makes an angle θ' with this axis.

If the conic is a parabola, we may take its equation as

$$a_{22}x_2{}^2 + a_{33}x_3{}^2 + 2a_{13}x_1x_3 = 0,$$

where $a_{22}\neq 0$, but a_{13} or a_{33} may or may not be zero.

Then (i) reduces to

$$\rho u_1' = a_{22}u_1, \quad u_2' = 0, \quad \rho u_3' = a_{22}u_3 - a_{13}u_1.$$

Therefore, for any $\mathbf{u}$, $\mathbf{u}'$ is perpendicular to the axis of the parabola and all lines in the pencil $a_{22}u_3 - a_{13}u_1 = ka_{22}u_1$ (where k is a constant) have the line $(1, 0, k)$ as their prime.

12. With respect to a parabola (degenerate or not), the lines of a pencil whose vertex is on the axis of the parabola have the same prime, and this is perpendicular to the axis of the parabola.   YUAN-JEN ROAN.

2133. *Notes on conics* 13. *The pedal property of the auxiliary circle.*

If $a/b = c/d$, then $a/c = b/d$. Geometrically, if the triangles OAB, OCD are similar, then also OAC is similar to OBD. If two circles cut in Q and O, and if one line through Q cuts the two circles again in A, B, and another line through Q cuts them again in C, D, the simple congruences

$$(OA, QA) \equiv (OC, QC), \quad (OB, QB) \equiv (OD, QD)$$

imply that the triangles OAB, OCD are similar, and the less obvious similarity of OAC to OBD can be regarded as a consequence.

If the tangent at a point P of a conic cuts the directrix in Z and if SY is the perpendicular from S to the tangent, the two circles $SZYX$, $SZPM$ are cut in Y, P by one line through Z and in X, M by another line through Z. Hence SYX is similar to SPM, and Y is on the Apollonian locus $|SY| = e |XY|$.

We may look at the similarity of the triangles OAB, OCD of our first paragraph in another way. Let two fixed circles cut in Q and O, and let a variable line through Q cut them again in A and B; then the variable triangle OAB, rotating round the fixed vertex O, has always the same shape. To take the corresponding view of the similarity of OAC, OBD, we must think of the two lines QAB, QCD as fixed lines, cut by a variable circle through the fixed points Q and O. This variable circle cuts the fixed lines in A and C, and it is the triangle OAC which maintains its shape. This form of the theorem is easy to understand if we think first of two parallel lines a, e cut by a variable line through O in points A and E; the ratio OA/OE is constant. If the line e with any range of points E_1, E_2, ... is rotated as a rigid system round O, so that the range becomes a range C_1, C_2, ... on a line c which cuts a in a point Q, the ratio $|OA|/|OC|$ and the angle between OA and OC are constant, and therefore the triangle OAC has the same shape for all positions of A. Also, since $(OA, OC) \equiv (e, c) \equiv (a, c)$, the circle OAC passes through Q, that is, the correlated points A, C may be regarded as the points in which the lines a, c are cut by a variable circle through O and Q. E. H. N.

2134. *On the generalised Vandermonde determinant.*

1. Vandermonde's determinant is

$$V_n = \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix} = \underset{i > k}{\varPi} (x_i - x_k),$$

where the x_i are elements of a commutative ring. Let us denote by V_k^n the determinant of n rows and columns:

$$\begin{vmatrix} 1 & x_1 & \dots & x_1^{k-1} & x_1^{k+1} & \dots & x_1^n \\ \vdots & & & \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^{k-1} & x_n^{k+1} & \dots & x_n^n \end{vmatrix},$$

the column of the kth powers being omitted. We have then $V_n^n = V_n$. These determinants occur in many parts of mathematics; we shall call them generalised Vandermonde determinants. The upper index which gives the order of the determinant will be left out in the rest of this Note.

It is well known that

$$V_k = V_n \cdot a_{n-k},$$

where a_{n-k} is the $(n-k)$th elementary symmetric function of the x_i, that is, the coefficient of t^k in the expansion of

$$\underset{i=1}{\overset{n}{\varPi}} (t + x_i).$$

A proof is given in Polya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, p. 302, no. 10. Another proof can be obtained from the Laplace expansion with respect to the first row of the determinant of order $n+1$:

$$\begin{vmatrix} 1 & -t & t^2 & \ldots & (-t)^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{vmatrix}.$$

The following proof is perhaps of interest, because it provides a simple illustration, suitable for scholarship work in schools, of the rule for the differentiation of a determinant the elements of which are functions of a variable.

Consider the determinant

$$D(t) = \begin{vmatrix} (t+x_1) & \ldots & (t+x_1)^n \\ \vdots & & \vdots \\ (t+x_n) & \ldots & (t+x_n)^n \end{vmatrix}.$$

By removing the factor $(t+x_i)$ from the ith row we see that

$$D(t) = \Pi(t+x_i) . \begin{vmatrix} 1 & t+x_1 & \ldots & (t+x_1)^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & t+x_n & \ldots & (t+x_n)^{n-1} \end{vmatrix}.$$

The latter determinant is the Vandermonde determinant of the quantities $t+x_1$, $t+x_2$, ... $t+x_n$, and since only differences enter into this, its value is independent of t and equal to V_n. Hence

$$D(t) = \Pi(t+x_i) . V_n.$$

Now the coefficient of t^k is, by Taylor's formula, $D^{(k)}(0)/k!$, where $D^{(k)}(t)$ denotes the kth derivative of $D(t)$ with respect to t. But when $D(t)$ is differentiated column by column with respect to t, we get

$$D'(t) = \begin{vmatrix} 1 & (t+x_1)^2 & \ldots & (t+x_1)^n \\ \vdots & \vdots & & \vdots \\ 1 & (t+x_n)^2 & \ldots & (t+x_n)^n \end{vmatrix} + \ldots + \ldots ,$$

where all the remaining $n-1$ determinants vanish because they have two proportional columns. Hence $D'(0)/1! = V_1$.

Similarly

$$D''(t) = \ldots + \begin{vmatrix} 1 & 2(t+x_1) & (t+x_1)^3 & \ldots & (t+x_1)^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 2(t+x_n) & (t+x_n)^3 & \ldots & (t+x_n)^n \end{vmatrix} + \ldots ,$$

where the only non-vanishing determinant is the one written out. Hence $D''(0)/2! = V_2$. It is now easy to see that the only non-vanishing determinant in $D^{(k)}(t)$ is

$$\begin{vmatrix} 1 & 2(t+x_1) & 3(t+x_1)^2 & \ldots & k(t+x_1)^{k-1} & (t+x_1)^{k+1} & \ldots & (t+x_1)^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & 2(t+x_n) & 3(t+x_n)^2 & \ldots & k(t+x_n)^{k-1} & (t+x_n)^{k+1} & \ldots & (t+x_n)^n \end{vmatrix},$$

so that $D^{(k)}(0)/k! = V_k$, which proves the result.

2. One of the simplest applications of the generalised Vandermonde determinants occurs in finding the condition that four points on the ellipse $x = a\cos\theta$, $y = b\sin\theta$ should be concyclic. Let θ_i, $i = 1, 2, 3, 4$, be the four values of the parameter. The determinantal condition for concyclic points is

$$| \; 1 \quad x_i \quad y_i \quad x_i^2 + y_i^2 \; | = 0, \quad i = 1, 2, 3, 4.$$

The substitution $\tan \frac{1}{2}\theta = t$ gives

$$| \ (1+t_i{}^2)^2 \quad 1 - t_i{}^4 \quad t_i(1+t_i{}^2) \quad a^2(1-t_i{}^2)^2 + 4b^2t_i{}^2 \ | = 0.$$

After simplification this becomes

$$| \ 1 \quad t_i \quad t_i{}^2 \quad t_i{}^4 \ | - | \ 1 \quad t_i{}^2 \quad t_i{}^3 \quad t_i{}^4 \ | = 0 ;$$

and according to the previous result, we have

$$\underset{i>k}{\Pi} (t_i - t_k) \cdot \{(t_1 + t_2 + t_3 + t_4) - (t_1 t_2 t_3 + \ldots)\} = 0.$$

This yields the condition

$$\tan \tfrac{1}{2}(\theta_1 + \theta_2 + \theta_3 + \theta_4) = 0$$

or

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 2k\pi.$$

3. The generalised Vandermonde determinants can also be treated as a special case of a type of determinant investigated by the author in the *Journal London Math. Soc.*, 24 (1949), 144-5. K. A. HIRSCH.

2135. *Tests for divisibility.*

In Note 1920 (*Gazette*, July 1946) Mr. S. Parameswaran has given tests for divisibility by numbers of the form $10^r + 1$, $10^r - 1$ and their factors. The following may be considered as extensions of these, and are easily proved.

1. General test for $10^n \cdot a + p$.

Mark off n digits from the right and test the difference between a times the right-hand period and p times the left-hand period.

Thus to test 5992`229 for divisibility by 79 (see table below) :

$$2 \times 5992 = 11`984 \qquad\qquad 3 \times 297 = 891$$
$$3 \times 229 = \ \ \ \ 687 \qquad\qquad 2 \times \ \ 11 = \ \ 22$$
$$\text{diff.} = \overline{11`297} \qquad\qquad\quad \text{diff.} = \overline{869} = 11 \times 79$$

and to test 32868171873 for divisibility by 2001 :

$$
\begin{array}{r}
32\ 868\ 171`873 \\
1\ 746 \\
\hline
32\ 866`425 \\
850 \\
\hline
32`016 \\
32 \\
\hline
0
\end{array}
$$

Thus 2001 is a 'factor and so are its subfactors 23 and 29.

2. General test for $10^n - p$.

Mark off into periods of n digits from the right. Multiply the left-hand period by p and add to its immediate neighbour. If a number of $n + 1$ digits is obtained as product or sum, treat this in the same way. Continue until a number of n digits is obtained after reaching the right-hand period.

Test 2068137 for divisibility by 97 $(= 10^2 - 3)$:

$$
\begin{array}{rrr}
2` & 06` & 81` \quad 37 \\
6 & 36 & 60 \\
\hline
12 & 17 & 97 \\
& 3 & \\
\hline
& 20 &
\end{array}
$$

$[06 + 3 \times 2 = 12 \ ; \quad 81 + 3 \times 12 = 117 \text{ written } 17 + 3 \times 1 = 20, \text{ etc.}]$

3. A test similar to that given in 1 above could be devised for $10^n \cdot a - p$, but it would be tedious in operation. When $p = 1$ it may be stated thus : Mark off n digits from the right. Multiply the number so formed by $a + 1$ and add the product to the number formed by the remaining digits. Repeat the process as often as may be necessary.

Test 412673 for divisibility by 499 :

$$
\begin{array}{r}
41\ 26`73 \\
3\ 65 \\
\hline
44`91 \\
455 \\
\hline
499
\end{array}
$$

4. The following table shows multiples of some small primes appropriate to be used as test divisors :

Primes	Multiples	Primes	Multiples
7, 11, 13	1001	53	901
13, 23	299	59	1003
17	799, 1003	67	201
19	399	73, 137	10001
23, 29	2001	79	3002
29, 31	899	89	801
37	999	127	8001
41, 61	10004	167	1002
43	301	101	9999
47	799		

199, 499, 599, 101, 401, 601, 701 are primes.

Two others may be given as curiosities. 100006 is a multiple of 31, and 130001 of 71.   B. A. SWINDEN.

2136. *Proofs of isosceles triangle theorems.*

Given a triangle ABC in which $AB = AC$; to prove $\angle C = \angle B$.
In the triangles ABC, ACB,

$$AB = AC, \quad AC = AB, \quad \angle A \text{ is common.}$$

Hence $\triangle ABC \equiv \triangle ACB$ (S.A.S.),

and therefore $\angle C = \angle B.$

A similar proof is available for the converse, using the A.A. corr. S. case of congruence.

I have tried this proof on Form IIc, and although it was not required to be learnt for reproduction it appeared to be appreciated by those capable of understanding formal geometry at all. The notion of a triangle being congruent to itself the other way round was readily accepted, but there was some initial scepticism about the legitimacy of using the same pair of sides twice.

This method avoids the distasteful construction and the explanation of why it is hypothetical. The theorem can be used, if desired, to prove the S.S.S. case of congruence, which is, after all, less obvious than the earlier cases. Finally, for the few who will become specialists, the idea of a pair of mutually corresponding elements is a preparation for involution as a special case of homography.   T. KNAPE SMITH.

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2137. *On the principle of proportional parts.*

There is a tendency for some pupils to use this principle, in numerical work, to give a result to more places of decimals than there are in the tabulated data ; this method may still be found in some textbooks. Below is calculated the probability of obtaining a correct figure in the $(n+1)$th decimal place when an intermediate value of a function is calculated from n-place tables.

Let y_1, y_2 be the correct values of the function for two successive values of the argument, and let $y_1 + \epsilon_1$, $y_2 + \epsilon_2$ be the tabulated values. These values are used to calculate the function at a point dividing the range in the ratio $\xi : 1 - \xi$; it is easily seen that the error in using the tabulated values for finding this is $\epsilon_2 \xi + \epsilon_1 (1 - \xi)$. (We are not, of course, concerned here with the approximation involved in using the principle due to the non-linear character of the function.) The problem is, then : what is the probability that $\epsilon_2 \xi + \epsilon_1 (1 - \xi)$ lies between $\pm \kappa$ when ϵ_1, ϵ_2 are known to lie between $\pm 10\kappa$? It is assumed that all values of ϵ_1, ϵ_2 are equally likely to occur.

If a simultaneous pair of values of ϵ_1, ϵ_2 is represented by a point of a plane by means of rectangular cartesian coordinates, we are concerned with points inside the square bounded by the lines $\epsilon_1 = \pm 10\kappa$, $\epsilon_2 = \pm 10\kappa$. The error corresponding to this pair of values is the value of the parameter λ of
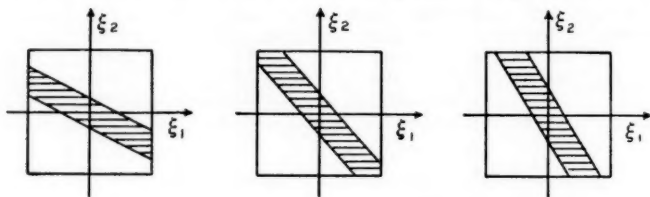
that member of the family of lines $\epsilon_2 \xi + \epsilon_1 (1 - \xi) = \lambda$ which passes through the point (ϵ_1, ϵ_2). The required probability p is the ratio of the area inside the square bounded by the lines $\lambda = \pm \kappa$ to that of the whole square. This ratio varies with ξ, the calculations being different according to the relation of the corners $(\pm 10\kappa, \mp 10\kappa)$ to the band bounded by the lines $\lambda = \pm \kappa$.

$$\text{(i) If } \xi > \tfrac{11}{20}, \qquad p = \frac{1}{10\xi}.$$

$$\text{(ii) If } \tfrac{9}{20} < \xi < \tfrac{11}{20}, \quad p = 1 - \frac{81}{400\xi(1 - \xi)}.$$

$$\text{(iii) If } \xi < \tfrac{9}{20}, \qquad p = \frac{1}{10(1 - \xi)}.$$

The method is most likely to give the correct answer when $\xi = \cdot 5$, for the mid-point of the range, when $p = \cdot 19$. If it is used indiscriminately for all values of ξ, the probability of obtaining the correct answer is $\int_0^1 p \, d\xi = \cdot 138$.

If the method is used only for the values $\cdot 1, \cdot 2, \ldots \cdot 9$ of ξ (*i.e.* for the first untabulated decimal place in the argument), the probability is the average of the 9 values of p for the values of ξ, and this is $\cdot 142$. Perhaps pupils would be less impressed by the method if they knew it would only afford the right answer once in every seven calculations.

One of the statements made above requires slight modification ; it is not true that if the correct and calculated values differ by less than κ, then they

agree in the $(n+1)$th decimal place. Suppose that 4-place tables are used, so that $\kappa = \cdot000005$, and consider the following pairs of values :

$$\begin{cases} \text{Correct} & - & \cdot470366 & \cdot470356 \\ \text{Calculated} & - & \cdot470364 & \cdot470364 \end{cases}$$

In the first case the difference is only $\cdot4\kappa$, but the values differ in the 5th place ; in the second it is $1\cdot6\kappa$, but they agree in the 5th place. Since the incidence of both cases is likely to occur an approximately equal number of times, the values for p given above are still valid. D. A. QUADLING.

2138. Rational plane quintics with three cusps.

The general plane quintic has 20 degrees of freedom. If we choose 6 arbitrary points A, B, C, P, Q, R and require the curve to have cusps at A, B, C and nodes at P, Q, R, we are imposing $3 \times 4 + 3 \times 3 = 21$ conditions, so there is not, in general, a quintic curve satisfying them. We must expect some restriction on the points if such a curve exists, and in this note I prove :

I. *If a quintic plane curve has cusps at A, B, C and nodes at P, Q, R, then the conics $ABCQR$, $ABCRP$, $ABCPQ$ have a common tangent.*

1. Let $A(1, 0, 0)$; $B(0, 1, 0)$; $C(0, 0, 1)$; $P(x_0, y_0, z_0)$; $Q(x_1, y_1, z_1)$; $R(x_2, y_2, z_2)$ be six points in the plane, no three collinear.

If we take PQR instead of ABC as triangle of reference, the coordinates of A, B, C may be taken to be :

$$\left.\begin{array}{c} A(X_0, \ X_1, \ X_2) \\ B(Y_0, \ Y_1, \ Y_2) \\ C(Z_0, \ Z_1, \ Z_2) \end{array}\right\} , \dots\dots\dots\dots\dots\dots\dots(1)$$

where X_0, Y_1, ... are the cofactors of x_0, y_1, ... in the matrix :

$$\begin{pmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} .$$

This follows from the substitution :

$$k\xi = X_0 x + Y_0 y + Z_0 z,$$
$$k\eta = X_1 x + Y_1 y + Z_1 z,$$
$$k\zeta = X_2 x + Y_2 y + Z_2 z,$$

which transforms PQR into the triangle of reference and A, B, C into the points (1).

2. (i) The general equation of a conic touching the sides of ABC is :

$$\lambda^2 x^2 + \mu^2 y^2 + \nu^2 z^2 - 2\mu\nu yz - 2\nu\lambda zx - 2\lambda\mu xy = 0,$$

or
$$\sqrt{\lambda x} \pm \sqrt{\mu y} \pm \sqrt{\nu z} = 0.$$

Hence the condition that a conic exists touching BC, CA, AB, and passing through P, Q, R is :

$$\Pi \begin{vmatrix} \sqrt{x_0} & \sqrt{y_0} & \sqrt{z_0} \\ \sqrt{x_1} & \pm\sqrt{y_1} & \pm\sqrt{z_1} \\ \sqrt{x_2} & \pm\sqrt{y_2} & \pm\sqrt{z_2} \end{vmatrix} = 0. \quad\dots\dots\dots\dots\dots\dots(2)$$

(The product is taken over all possible alternative signs, so that the left-hand side is a polynomial in the x, y, z.)

For shortness we write this condition as :

$$\| \sqrt{x} \| = 0. \quad\dots\dots\dots\dots\dots\dots\dots\dots\dots(3)$$

Using (1), we deduce :

The condition for a conic touching PQ, QR, RP, and passing through A, B, C is :

$$\| \sqrt{X} \| = 0 \quad\text{.....................................(4)}$$

(where x_0, y_1, ... in (2) are replaced by X_0, Y_1, ...)

(ii) Condition for the conics $ABCQR$, $ABCRP$, $ABCPQ$ to have a common tangent.

Apply the quadratic transformation $(x, y, z) \to \left(\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}\right)$. (We shall denote this transformation by S in what follows.) Let P', Q', R' be the points corresponding to P, Q, R.

The three conics $ABCQR$, etc., correspond to the lines $Q'R'$, $R'P'$, $P'Q'$, and the common tangent corresponds to a conic through A, B, C touching these three lines.

Hence (4) holds with x_0, y_1, ... replaced by $\dfrac{1}{x_0}$, $\dfrac{1}{y_1}$, ...

But
$$\frac{1}{y_1 z_2} - \frac{1}{y_2 z_1} = \frac{y_2 z_1 - y_1 z_2}{x_0} \cdot \frac{x_0 x_1 x_2}{x_1 y_1 z_1 \cdot x_2 y_2 z_2}$$
$$= -\frac{X_0}{x_0} \cdot \frac{x_0 x_1 x_2}{x_1 y_1 z_1 \cdot x_2 y_2 z_2} .$$

Since no three of our points are collinear, we can cancel factors $\sqrt{(x_0 x_1 x_2)}$ from the rows and $\dfrac{1}{\sqrt{(x_1 y_1 z_1 x_2 y_2 z_2)}}$ from the columns of the determinants, and the required condition takes the form :

$$\| \sqrt{(X/x)} \| = 0. \quad\text{.................................(5)}$$

(iii) Applying S to (3), we have :
Condition for quartic through P, Q, R with cusps at A, B, C is :

$$\| 1/\sqrt{x} \| = 0 ;$$

so : condition for quartic through A, B, C with cusps at P, Q, R is :

$$\| 1/\sqrt{X} \| = 0. \quad\text{......................................(6)}$$

Applying S to (6) :
Condition for quintic with double points at A, B, C, cusps at P, Q, R is :

$$\| \sqrt{(x/X)} \| = 0 ;$$

so : condition for quintic with cusps at A, B, C, double points at P, Q, R is :

$$\| \sqrt{(X/x)} \| = 0. \quad\text{.....................................(7)}$$

Comparison of (5) and (7) proves the result I.

3. We can obtain another result by applying the transformation S to (7), getting :
Condition for quartic with double points at A, B, C, which passes through P, Q, R and also touches QR, RP, PQ, is :

$$\| \sqrt{X} \| = 0,$$

and this is the same as (4).

We have thus proved :

II. *If the triangle PQR is both inscribed in and circumscribed to a quartic with nodes at A, B, C, then a conic exists passing through P, Q, R and touching BC, CA, AB.*   A. M. MACBEATH.

2139. *On converse theorems of summability* * *: addendum.*

It is perhaps worth while recording the fact that Theorem B can be expressed in a form which involves, instead of the restriction (b) on $|a_n|$, a restriction on $|a_n|/|a_{n-1}|$. In this new form, stated below, the theorem invites comparison with a sequence of ratio-tests (including Kummer's) for the convergence of series of positive terms discussed by me elsewhere.†

Theorem B_1. *If the series* Σa_n *is summable-$R(\lambda_n, 1)$ to s, the condition*

$$\frac{1}{\lambda_n - \lambda_{n-1}} \log \frac{|a_n|(\lambda_{n-1} - \lambda_{n-2})}{|a_{n-1}|(\lambda_n - \lambda_{n-1})} \leqslant -\frac{1}{\lambda_{n-1}} \quad\dots\dots\dots\dots(b_1)$$

ensures the convergence of the series to the sum s.

Proof. We have, in consequence of (b_1), the inequalities

$$\log \frac{|a_\nu|(\lambda_{\nu-1} - \lambda_{\nu-2})}{|a_{\nu-1}|(\lambda_\nu - \lambda_{\nu-1})} \leqslant -\int_{\lambda_{\nu-1}}^{\lambda_\nu} \frac{dx}{x};$$

summing these from $\nu = n_0 + 1$ to $\nu = n$ and adding the sum to

$$0 \leqslant \int_x^{\lambda_n} \frac{dx}{x} \quad (\lambda_{n-1} \leqslant x \leqslant \lambda_n),$$

we obtain

$$\frac{|a_n|}{\lambda_n - \lambda_{n-1}} \leqslant \frac{|a_0|}{\lambda_{n_0} - \lambda_{n_0-1}} \frac{\lambda_{n_0}}{x}, \quad (\lambda_{n-1} \leqslant x \leqslant \lambda_n),$$

or, after integrating both sides from λ_{n-1} to λ_n,

$$|a_n| \leqslant H \log (\lambda_n/\lambda_{n-1}),$$

where

$$H = |a_{n_0}| \lambda_{n_0}/(\lambda_{n_0} - \lambda_{n_0-1}).$$

Hence, writing

$$s_n = \sum_{\nu=1}^{n} a_\nu,$$

we see that

$$|s_{m+p+1} - s_{m+1}| \leqslant |a_{m+2}| + \dots + |a_{m+p+1}|$$
$$\leqslant H \log (\lambda_{m+p+1}/\lambda_{m+1}).$$

Thus, if we suppose (as we may without loss of generality) that s_n is real, both s_n and $-s_n$ satisfy condition $(4a)$ in the proof of Theorem A ; and we reach the desired conclusion in the form

$$\overline{\lim_{n \to \infty}}\, s_n = s = \varliminf_{n \to \infty} s_n.$$

C. T. R.

2140. *On division by* $(\alpha)_k$.

1. *Notation.*

(i) The digit α, $1 \leqslant \alpha \leqslant 9$, repeated k times, is denoted by $(\alpha)_k$.

(ii) $[x]$ denotes the integral part and $\{x\}$ the fractional part of x.

(iii) $N = a_1 a_2 \dots a_n$ where a_2, a_3, $\dots$ a_n each contain k digits and a_1 may contain k digits or less.

(iv) $Q(N) = \left[\dfrac{N}{10^k}\right] + \left[\dfrac{N}{10^{2k}}\right] + \left[\dfrac{N}{10^{3k}}\right] + \dots$

$= a_1 a_2 \dots a_{n-1} + a_1 a_2 \dots a_{n-2} + \dots + a_2 a_1 + a_1$;

$R(N) = a_1 + a_2 + \dots + a_n.$

* *Math. Gazette*, XXX (1946), pp. 272–6.

† *Journ. Indian Math. Soc.* (New Series), 3 (1938), p. 123, Theorem 4 ; *Math. Gazette*, XXIII (1939), p. 461, Remark.

2. In this note we establish formulae for the quotient and remainder when any number N of type (iii) is divided by $(\alpha)_k$:

$$\text{quotient} = [9Q(N)/\alpha + R(N)/(\alpha)_k],$$
$$\text{remainder} = \{9Q(N)/\alpha + R(N)/(\alpha)_k\}(\alpha)_k.$$

For these results we require the following lemmas.

Lemma I. On dividing N by $(9)_k$ the remainder is $\{R(N)/(9)_k\}(9)_k$.

For
$$N = a_1(10^k)^{n-1} + a_2(10^k)^{n-2} + \ldots + a_n$$
$$\equiv a_1 + a_2 + \ldots + a_n \quad (\text{mod } (9)_k).$$

Lemma II. On dividing N by $(9)_k$ the quotient is $Q(N) + [R(N)/(9)_k]$.

$$[N/(9)_k] = [(a_1 a_2 \ldots a_{n-1} \cdot 10^k + a_n)/(9)_k]$$
$$= a_1 a_2 \ldots a_{n-1} + [(a_1 a_2 \ldots a_{n-1} + a_n)/(9)_k].$$

Repeating the argument we get the result *or*

$$\frac{N}{(9)_k} = \frac{a_1(10^{k(n-1)} - 1) + a_2(10^{k(n-2)} - 1) + \ldots + a_n(1 - 1) + a_1 + a_2 + \ldots + a_n}{(10^k - 1)}$$
$$= a_1(10^{k(n-2)} + 10^{k(n-3)} + \ldots + 1) + a_2(10^{k(n-3)} + \ldots + 1) + \ldots$$
$$\qquad + a_{n-2}(10^k + 1) + a_{n-1} + [(a_1 + a_2 + \ldots + a_n)/(9)_k]$$
$$= a_1 a_2 \ldots a_{n-1} + a_1 a_2 \ldots a_{n-2} + \ldots + a_1 a_2 + a_1 + [R(N)/(9)_k].$$

3. Let x be the quotient and y the remainder on dividing N by $(\alpha)_k$.

Then
$$N = x(\alpha)_k + y = Q(N)(9)_k + R(N).$$

Thus
$$y \equiv Q(N)(9)_k + R(N) \quad (\text{mod } (\alpha)_k)$$
$$= \{Q(N)(9)_k/(\alpha)_k + R(N)/(\alpha)_k\} \cdot (\alpha)_k$$
$$= \{9Q(N)/\alpha + R(N)/(\alpha)_k\} \cdot (\alpha)_k.$$
$$x = [Q(N) \cdot (9)_k/(\alpha)_k + R(N)/(\alpha)_k]$$
$$= [9Q(N)/\alpha + R(N)/(\alpha)_k].$$

This note helps us to apply the " tests for divisibility " given in Note 1920, *Mathematical Gazette*, XXX, No. 290.   S. PARAMESWARAN.

2141. *Some series for log 2.*

I premise the following known * results, where γ is Euler's constant and $s_r = 1^{-r} + 2^{-r} + 3^{-r} + \ldots$:

$$\lim_{x \to 1} \frac{d}{dx} \log \Gamma(x) = -\gamma, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{d^2}{dx^2} \log \Gamma(x) = \frac{1}{x^2} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \ldots, \quad \ldots\ldots\ldots\ldots\ldots(2)$$

$$\log \Gamma(1+x) = -\gamma x + s_2 x^2/2 - s_3 x^3/3 + \ldots, \quad \ldots\ldots\ldots\ldots\ldots(3)$$

$$\log \Gamma(1+x) = -\log(1+x) + (1-\gamma)x + (s_2-1)x^2/2 - (s_3-1)x^3/3 + \ldots \ldots(4)$$

Integrating (2) between the limits x and 1, we have

$$\frac{d}{dx} \log \Gamma(x) + \gamma = -\left\{ \left(\frac{1}{x} - 1\right) + \left(\frac{1}{x+1} - \frac{1}{2}\right) + \ldots + \left(\frac{1}{x+n} - \frac{1}{n+1}\right) + \ldots \right\}, \ldots(2a)$$

the series converging for all values of x save negative integers ; and by differentiating (3) and (4) :

* *Vide* Schlömilch, *Kompendium*, II. Professor T. J. I'A. Bromwich has extended the region of convergence of series (4) in the text from $|x| \leqslant 1$ to $|x| \leqslant 2$.

$$\frac{d}{dx}\log \Gamma(x) + \gamma = s_2(x-1) - s_3(x-1)^2 + s_4(x-1)^3 - \dots, \quad \dots\dots\dots(3a)$$

and $\quad \dfrac{d}{dx}\log \Gamma(x) + \gamma = (x-1)/x + (s_2-1)(x-1) - (s_3-1)(x-1)^2 + \dots, \quad \dots(4a)$

both series converging for $|x-1| < 1$.

Thus so long as the series involved converge

$$-\left\{\left(\frac{1}{x}-1\right) + \left(\frac{1}{x+1}-\frac{1}{2}\right) + \left(\frac{1}{x+2}-\frac{1}{3}\right) + \dots\right\}$$

$$= s_2(x-1) - s_3(x-1)^2 + s_4(x-1)^3 - \dots$$

$$= (x-1)/x + (s_2-1)(x-1) - (s_3-1)(x-1)^2 + \dots. \quad \dots\dots(5)$$

The first equality does not hold for $|x-1| = 1$.

If we now put $x = \frac{1}{2}$ and remove the brackets in the first series of (5), as is permissible, we obtain

$$2\log 2 = \sum_{r=2}^{\infty} s_r/2^{r-1}$$

$$= 1 + \sum_{r=2}^{\infty} (s_r-1)/2^{r-1}. \quad \dots\dots\dots\dots\dots(6)$$

If we use for $\log \Gamma(1+x)$ the series

$$\tfrac{1}{2}\log\frac{\pi x}{\sin \pi x} + \tfrac{1}{2}\log\frac{1-x}{1+x} + (1-\gamma)x - (s_3-1)\frac{x^3}{3} - (s_5-1)\frac{x^5}{5} - \dots,$$

and, after differentiation, put $x = -\frac{1}{2}$, we have

$$2\log 2 = \tfrac{4}{3} + \sum_{1}^{\infty}(s_{2r+1}-1)/2^{2r}. \quad \dots\dots\dots\dots\dots\dots(7)$$

Further, (4) is equivalent to

$$\log \Gamma(2+x) = (1-\gamma)x + (s_2-1)x^2/2 - (s_3-1)x^3/3 + \dots;$$

whence

$$\tfrac{1}{2}\{\log \Gamma(2+x) + \log \Gamma(2-x)\} = (s_2-1)x^2/2 + (s_4-1)x^4/4 + \dots,$$

for $|x| < 2$. Putting $x = 1$ we have

$$\log 2 = \sum_{1}^{\infty}(s_{2r}-1)/r. \quad \dots\dots\dots\dots\dots\dots\dots(8)$$

Many other particular results may be obtained by this method. For (8) compare *Proc. Edin. Math. Soc.*, XXVIII, p. 56.   B. A. SWINDEN.

2142. *A property of bi-variate normal distributions.*

This note deals with a property of bi-variate normal statistical distributions of distribution function

$$\frac{1}{2\pi k\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left(x^2 + \frac{y^2}{k^2}\right)\right\},$$

so that σ is the root mean square in the x direction and $k\sigma$ is the same quantity in the y direction.

Let f be that fraction of the total number of observations of modulus $\sqrt{(x^2+y^2)}$ less than R_f, and P_f the ratio of R_f to the root mean square value, which is $\sigma\sqrt{(1+k^2)}$.

Now take the one-variable distribution of function

$$\frac{1}{\sigma\sqrt{(2\pi)}} \exp\left(-x^2/2\sigma^2\right).$$

In this case f is that fraction of the total number of observations at a distance less than R_f from the centre, and P_f is the ratio of R_f to the root mean square value σ. It was noticed by C. H. B. Priestley in the course of some work on the errors of wind measurement published by the Meteorological Office that for f equal to 80 per cent., the value of P_f was $1\cdot27$ in the case of the circular distribution, for which $k=1$, and to $1\cdot28$ for the one-variable distribution. He conjectured that the ratio was nearly $1\cdot27$ for the elliptic distributions intermediate between the circular and scalar distributions.

This note renders the conjecture more precise and shows that if P_f is so chosen that f is equal in both the scalar and circular cases, then, for the same value of P_f, f will not deviate by more than 2 per cent. in the intermediate elliptic distributions.

In the general elliptic case the fraction of observations inside the circle centre the mean point and radius R_f is given by

$$f = \frac{1}{2\pi k\sigma^2} \int_0^{2\pi} d\theta \int_0^{R_f} \exp\left\{ -\frac{r^2}{2\sigma^2}\left(\cos^2\theta + \frac{\sin^2\theta}{k^2}\right)\right\} r\, dr. \quad\ldots\ldots\ldots\ldots(1)$$

Then $P_f = R_f/\sigma\sqrt{(1+k^2)}$ and the relation between f and P_f is

$$f = \frac{2}{\pi k} \int_0^{P_f\sqrt{(1+k^2)}} R\, dR \int_0^{\pi/2} \exp\left\{ -\frac{R^2}{2k^2}(k^2\cos^2\theta + \sin^2\theta)\right\} d\theta, \quad\ldots\ldots(2)$$

in which the substitution $r = R\sigma$ has been used.

In the circular case, $k=1$, this reduces to

$$f = 1 - \exp\left(-P_f^2\right). \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

In the scalar case, $k=0$,

$$f = \frac{2}{\sigma\sqrt{(2\pi)}} \int_0^{R_f} \exp\left(-x^2/2\sigma^2\right) dx, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

and $P_f = R_f/\sigma$, so

$$f = \frac{2}{\sqrt{\pi}} \int_0^{P_f/\sqrt{2}} \exp\left(-u^2\right) du = \mathrm{erf}\,(P_f/\sqrt{2}). \quad\ldots\ldots\ldots\ldots\ldots(5)$$

By putting $\sin\theta = kt$ in (2) and making k tend to zero, it is readily shown that the expression (2) for f tends to $\mathrm{erf}\,(P_f/\sqrt{2})$ as k tends to zero. We now choose P_f so that

$$1 - \exp\left(-P_f^2\right) = \mathrm{erf}\,(P_f/\sqrt{2}).$$

The required root of this equation in P_f is $1\cdot24$, and this is the only root apart from zero and infinity. The corresponding value of f in both the circular and scalar distributions is $78\cdot5$ per cent.

Now, re-arranging formula (2), it may be written

$$f = \frac{2}{\pi k} \int_0^{P_f\sqrt{(1+k^2)}} \exp\left\{ -\frac{R^2(k^2+1)}{4k^2}\right\} R\, dR \int_0^{\pi/2} \exp\left\{ \frac{R^2}{4k^2}(1-k^2)\cos 2\theta\right\} d\theta,$$

which, using the relation

$$\pi I_0(x) = \int_0^\pi \exp\left(x\cos\theta\right) d\theta,$$

may be written

$$f = \frac{1}{k} \int_0^{P_f\sqrt{(1+k^2)}} \exp\left\{ -\frac{R^2(k^2+1)}{4k^2}\right\} I_0\left\{ R^2\left(\frac{1-k^2}{4k^2}\right)\right\} R\, dR.$$

Writing
$$(1/k) + k = 2 \cosh \alpha = 2c,$$
$$(1/k) - k = 2 \sinh \alpha = 2s,$$

we find, putting $z = R^2/2k$,

$$f = \int_0^{P_f^2 c} e^{-cz} I_0(sz)\, dz. \quad \text{................................(6)}$$

Differentiating (6) with respect to α and noting that $\dfrac{df}{dk} = -\dfrac{1}{k}\dfrac{df}{d\alpha}$, we find

$$\frac{df}{dk} = -(P_f^2/k)\exp(-P_f^2 c^2)\{sI_0(scP_f^2) - cI_1(scP_f^2)\}. \quad \text{............(7)}$$

At $k = 1$, $df/dk = 0$, and, near $k = 1$, df/dk is dominated by $sI_0(scP_f^2)$, so that df/dk is negative. At $k = 0$, df/dk is zero, and using the asymptotic form of the Bessel functions, namely,

$$I_n(z) = \frac{e^z}{\sqrt{(2\pi z)}}\left\{1 - \frac{4n^2 - 1}{8z} + ...\right\},$$

it follows that df/dk is positive near $k = 0$. Hence df/dk must vanish for an odd number of values of k between 0 and 1. It is found by trial that it vanishes for one value only, namely, $k = 0.457$. Thus f has a maximum value for $k = 0.457$.

Evaluating f numerically for $k = 0.457$, $P_f = 1.24$, the value 80.2 per cent. is found and this gives the maximum deviation, for values of k between 0 and 1, from the value 78.5 per cent. which holds for the scalar and circular cases.

Hence we can derive a rule, necessary but not sufficient, for testing whether a bi-variate distribution is normal, as follows : find the percentage number of observations inside the circle centred at the mean position and of radius 1.24 times the mean square value. If the distribution is normal the percentage will be very close to 80 per cent.

This note is submitted by permission of the Director, Meteorological Office, Air Ministry.   G. A. BULL and E. KNIGHTING.

2143. *Mental multiplication.*

Some books of aids to mental arithmetic give a device for multiplying together two numbers having the same two digits, with complementary units digits. Those who are interested in this kind of thing may like to see this extension to any two numbers of two digits with complementary units digits :

Multiply together these multiples of 10 which are respectively just less than the smaller number and just larger than the greater number : add to this the product of the differences from the first of these of the two given numbers.

Taking an example, 27×83, as awkward as I could find, we proceed :

(a) $20 \times 90 = 1800.$
(b) $(27 - 20) \times (83 - 20) = 7 \times 63 = 441.$
(c) $1800 + 441 = 2241.$

                                         C. DUDLEY LANGFORD.

2144. *Some theorems on concurrence and collinearity.*

We have, first, the well-known theorem of elementary geometry :

I. The bisectors of the angles of a triangle are concurrent.

If, however, the angles are trisected, this simple property of concurrence appears to be lost, since the incentre is now replaced by the well-known Morley triangle. But if each vertex of the triangle is joined to the corresponding vertex of the Morley triangle, the three lines so drawn are found to be

I

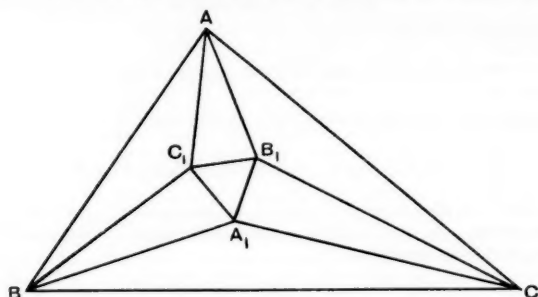concurrent. Thus, in Fig. 1, let A_1, B_1, C_1 be the Morley triangle obtained by

FIG. 1.

trisecting the angles ABC, BCA, CAB of the triangle ABC by means of the lines AC_1, AB_1, etc. Then the three lines AA_1, BB_1, CC_1 are concurrent.

But this property does not appear to be peculiar to trisected angles of a triangle. Indeed, as first proved, the theorem could be enunciated as follows:

II. If X is a point inside a triangle such that

$$\angle XBC = k \cdot \angle ABC, \quad \angle XCB = k \cdot \angle ACB,$$

and if points Y and Z are similarly defined with respect to C, A and A, B respectively, then AX, BY, CZ are concurrent.

It is now found that, in the proof of this theorem, we could go further and substitute the following more general theorem:

IIa. If X, Y, Z are three points inside a triangle ABC such that

$$\angle XBC = \angle ZBA = \beta, \quad \angle XCB = \angle YCA = \gamma, \quad \angle YAC = \angle ZAB = \alpha,$$

then the three lines AX, BY, CZ are concurrent.

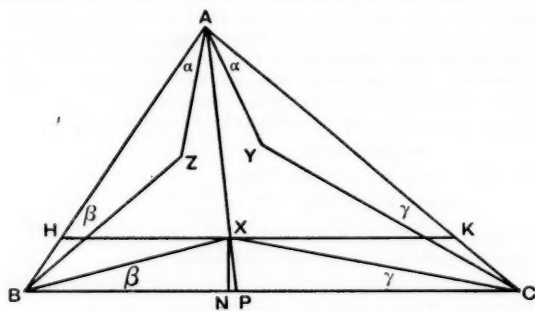The following simple proof is due to Mr. R. C. Lyness.

FIG. 2.

Join AX and produce to meet BC in P. Draw XN perpendicular to BC, and through X draw HXK parallel to BC to meet AB in H and AC in K. Then we have at once

$$HX = XN \ (\cot \beta - \cot B),$$

$$XK = XN \ (\cot \gamma - \cot C).$$

Also $\qquad BP/PC = HX/XK,$

whence $\qquad BP/PC = (\cot \beta - \cot B)/(\cot \gamma - \cot C).$

In a similar way we have

$$CQ/QA = (\cot \gamma - \cot C)/(\cot \alpha - \cot A),$$

and $\qquad AR/RB = (\cot \alpha - \cot A)/(\cot \beta - \cot B),$

where Q and R are points in AC, AB where BY and CZ cut these two lines. The result now follows at once by Ceva's theorem.

But this theorem now shows that there is a second point of concurrence in the triangle. For, clearly, if we produce the lines BZ and CY to meet in X', AY and BX to meet in Z', and AZ and CX to meet in Y', then it follows by the theorem just proved that AX', BY', CZ' must also be concurrent.
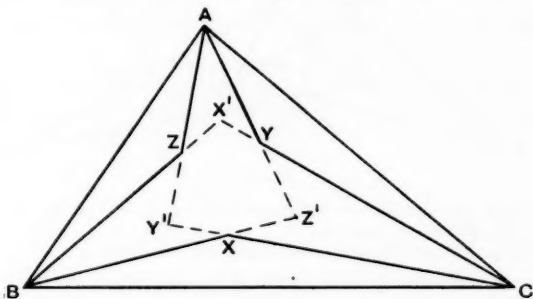

Fig. 3.

Our figure now consists of the original triangle ABC together with a hexagon $XY'ZX'YZ'$, as shown in Fig. 3. This hexagon has the further property of having the diagonals XX', YY', ZZ' concurrent, but for brevity we omit the proof of this result. We have thus three points of concurrence in the triangle ABC :

(a) AX, BY, CZ concurrent in I, say ;

(b) AX', BY', CZ' concurrent in J, say ;

(c) XX', YY', ZZ' concurrent in K, say.

These three points, I, J, K, are collinear, but we omit the proof.

We now have the general theorem :

III. If, from the three vertices A, B, C of a triangle ABC, pairs of lines are drawn, each pair inside its respective angle, such that each member of any pair is equally inclined to the arms of the angle enclosing that pair, and if the corresponding lines from the vertices meet in pairs in X, Y, Z, and also in X', Y', Z', then

(i) AX, BY, CZ are concurrent in a point I ;

(ii) AX', BY', CZ' are concurrent in a point J ;

(iii) XX', YY', ZZ' are concurrent in a point K ;

(iv) I, J, K are collinear.

We are thus led to consider a hexagon whose diagonals are concurrent, and we ask : If the diagonals XX', YY', ZZ' of a hexagon $XY'ZX'YZ'$ are

concurrent, what happens if the sides of such a hexagon be produced? This leads to a problem, which I call the three-hexagon problem, which shows that the incidences of Theorem III above are purely projective.

IV. If the diagonals X_1X_2, Y_1Y_2, Z_1Z_2 of a hexagon $X_1Y_2Z_1X_2Y_1Z_2$ are concurrent in a point I, and the sides X_1Y_2, Y_1X_2 meet in Z, Z_1X_2 and X_1Z_2 in Y, Y_1Z_2 and Z_1Y_2 in X, forming the hexagons $XY_1ZX_1YZ_1$ and $XY_2ZX_2YZ_2$, then the diagonals XX_1, etc., of the second hexagon are concurrent in a point J, the diagonals XX_2, etc., of the third are concurrent in a point K, and I, J, K are collinear.
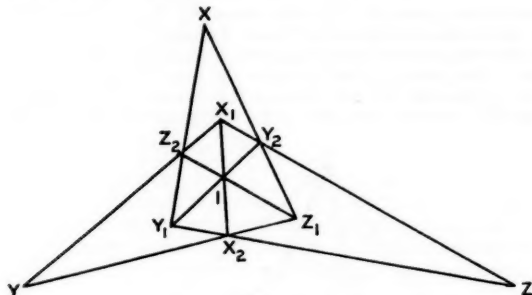

FIG. 4.

Take $X_1Y_1Z_1$ as triangle of reference and let $X_1 \equiv (1, 0, 0)$, $Y_1 \equiv (0, 1, 0)$, $Z_1 \equiv (0, 0, 1)$. Then $I \equiv (1, 1, 1)$.

Since X_1X_2, Y_1Y_2, Z_1Z_2 are concurrent, we can take

$$X_2 \equiv (\alpha, 1, 1), \quad Y_2 \equiv (1, \beta, 1), \quad Z_2 \equiv (1, 1, \gamma).$$

We can now show easily that

$$XY_1 \equiv \gamma x - z = 0, \quad XZ_1 \equiv \beta x - y = 0, \quad \quad \quad (1)$$
$$YZ_1 \equiv \alpha y - x = 0, \quad YX_1 \equiv \gamma y - z = 0, \quad \quad \quad (2)$$
$$ZX_1 \equiv \beta z - y = 0, \quad ZY_1 \equiv \alpha z - x = 0. \quad \quad \quad (3)$$

Equations (1), (2), (3) now give

$$X \equiv (1, \beta, \gamma), \quad Y \equiv (\alpha, 1, \gamma), \quad Z \equiv (\alpha, \beta, 1).$$

Also it is now easy to see that

$$XX_1 \equiv \gamma y - \beta z = 0, \quad \quad \quad \quad \quad (4)$$
$$YY_1 \equiv \beta x - \alpha y = 0, \quad \quad \quad \quad \quad (5)$$
$$ZZ_1 \equiv \alpha z - \gamma x = 0. \quad \quad \quad \quad \quad (6)$$

Since each of these equations is a linear combination of the others, it follows that XX_1, YY_1, ZZ_1 are concurrent in the point $(\alpha, \beta, \gamma) \equiv J$.

Next we have

$$XX_2 \equiv \begin{vmatrix} x & y & z \\ \alpha & 1 & 1 \\ 1 & \beta & \gamma \end{vmatrix} = 0, \quad YY_2 \equiv \begin{vmatrix} x & y & z \\ 1 & \beta & 1 \\ \alpha & 1 & \gamma \end{vmatrix} = 0,$$

$$ZZ_2 \equiv \begin{vmatrix} x & y & z \\ 1 & 1 & \gamma \\ \alpha & \beta & 1 \end{vmatrix} = 0;$$

that is

$$XX_2 \equiv x(\gamma - \beta) + y(1 - \gamma\alpha) + z(\alpha\beta - 1) = 0,$$
$$YY_2 \equiv x(\beta\gamma - 1) + y(\alpha - \gamma) + z(1 - \alpha\beta) = 0,$$
$$ZZ_2 \equiv x(1 - \beta\gamma) + y(\gamma\alpha - 1) + z(\beta - \alpha) = 0.$$

These equations are satisfied by the point

$$(x, y, z) \equiv (\alpha + 1, \beta + 1, \gamma + 1) \equiv K,$$

and thus the diagonals XX_2, YY_2, ZZ_2 are concurrent in K.

Finally, the line IJ is

$$\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{vmatrix} = 0,$$

and since

$$\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{vmatrix} = \begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ \alpha+1 & \beta+1 & \gamma+1 \end{vmatrix} = 0,$$

it follows that K is also on IJ, and thus the points I, J, K are collinear.

E. J. HOPKINS.

2145. *A note on compound determinants.*

Whittaker [*] has shown that certain compound determinants can be expressed as non-compound determinants. If $A(r, s)$ denotes an $r \times s$ array, Whittaker's non-compounds appear in the form :

$$\begin{vmatrix} A_1(m, m+1) & & & \\ & A_2(m, m+1) & & \\ & & \ddots & \\ & & & A_m(m, m+1) \\ B(m, m+1) & B(m, m+1) & \ldots & B(m, m+1) \end{vmatrix} \quad \ldots\ldots\ldots\ldots(1)$$

By Laplace expansion this determinant can be shown to be equal to a compound determinant either of the mth or $(m+1)$th order. The compounds which arise depend on the $m+1$ minors of order m contained in an m by $m+1$ array. The case treated here is the non-compound which arises from the $\frac{1}{2}m(m+1)$ minors of order $m-1$ which can be selected from an m by $m+1$ array. It appears that the determinant

$$\begin{vmatrix} A_1(2, n) & & & & \\ & A_2(2, n) & & & \\ \vdots & \vdots & & & A_{n-1}(2, n) \\ B(n-1, n) & B(n-1, n) & & & \\ & & B(n-1, n) & B(n-1, n) & \\ & & & B(n-1, n) & \\ & & & & B(n-1, n) \end{vmatrix}, \quad \ldots\ldots\ldots(II)$$

where

$$A_s(2, n) \equiv \begin{bmatrix} a_{1s} & a_{2s} & \ldots & a_{ns} \\ b_{1s} & b_{2s} & \ldots & b_{ns} \end{bmatrix},$$

$$B(n-1, n) \equiv \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & & & \\ c_{n-1,1} & c_{n-1,2} & \ldots & c_{n-1,n} \end{bmatrix},$$

* Whittaker, E. T., *Proc. Ed. Math. Soc.*, XXXVI ; also Muir, *Quart. Journ.*, xlviii.

can be expressed either as (a) a compound of order $\dfrac{n(n-1)}{2}$ whose elements are of order 2, or (b) a compound of order $n-1$ whose elements are of order n.

When $n=3$ the non-compound (II) is a special case of (I). When $n=4$ we have :

$$\begin{vmatrix} a_1a_2a_3a_4 \\ b_1b_2b_3b_4 \\ & c_1c_2c_3c_4 \\ & d_1d_2d_3d_4 \\ & & e_1e_2e_3e_4 \\ & & f_1f_2f_3f_4 \\ x_1x_2x_3x_4 & x_1x_2x_3x_4 \\ y_1y_2y_3y_4 & y_1y_2y_3y_4 \\ z_1z_2z_3z_4 & z_1z_2z_3z_4 \\ & x_1x_2x_3x_4 & x_1x_2x_3x_4 \\ & y_1y_2y_3y_4 & y_1y_2y_3y_4 \\ & z_1z_2z_3z_4 & z_1z_2z_3z_4 \end{vmatrix}$$

$$= -\mid \ \mid a_1b_2 \mid , \ \mid c_1d_3 \mid , \ \mid e_1f_4 \mid , \ \mid x_2y_3 \mid , \ \mid x_2z_4 \mid , \ \mid y_3z_4 \mid \ \mid \quad \ldots\ldots\ldots(a)$$

$$= -\mid \ \mid xyab \mid , \quad \mid xzcd \mid , \quad \mid yzef \mid \ \mid . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(b)$$

To prove (a) we notice first of all that the cofactor of a term like

$$\mid a_r b_s \mid . \mid c_r d_s \mid \quad (\text{or} \ \mid a_r b_s \mid . \mid e_r f_s \mid, \ \text{etc.}) \ \text{is zero.}$$

For in the minor we need only subtract columns and apply 3×3 Laplace expansion ; this feature is common to all orders of (II).

Considering the cofactor of any other three second order minors from the first six rows, for example, the cofactor of $\mid a_3 b_4 \mid . \mid c_1 d_2 \mid . \mid e_1 f_3 \mid$, we find

$$-\begin{vmatrix} x_1x_2 & x_3x_4 \\ y_1y_2 & y_3y_4 \\ z_1z_2 & z_3z_4 \\ & x_3x_4 & x_2x_4 \\ & y_3y_4 & y_2y_4 \\ & z_3z_4 & z_2z_4 \end{vmatrix} = -\begin{vmatrix} x_1x_4 & x_2x_3 \\ y_1y_4 & y_2y_3 \\ z_1z_4 & z_2z_3 \\ & x_2x_3 & x_2x_4 \\ & y_2y_3 & y_2y_4 \\ & z_2z_3 & z_2z_4 \end{vmatrix}$$

$$= -\begin{vmatrix} x_1y_1z_1 \\ x_4y_4z_4 \\ & x_2y_2z_2 \\ & x_4y_4z_4 \\ x_2y_2z_2 & x_2y_2z_2 \\ x_3y_3z_3 & x_3y_3z_3 \end{vmatrix} = -\mid \ \mid x_1y_4 \mid , \ \mid x_2z_4 \mid , \ \mid y_2z_3 \mid \ \mid$$

using Whittaker's form (I),

which agrees with the required cofactor in (a).

To prove (b) we write the transpose of $[a_1a_2a_3a_4]$ as a'. Then we consider

$$= \begin{vmatrix} a'b' & \ldots & \ldots & x'y'z' & & \ldots\ldots \\ \ldots & c'd' & \ldots & x'y'z' & x'y'z' & \\ \ldots\ldots & e'f' & & \ldots\ldots & x'y'z' \\ x'y'z' & & \ldots\ldots & a'b' & & \ldots\ldots \\ x'y'z' & x'y'z' & \ldots & c'd' & \ldots & \\ \ldots\ldots & & x'y'z' & & \ldots\ldots & e'f' \end{vmatrix}$$

$$= -\mid \ \mid x'y'a'b' \mid , \ \mid x'z'c'd' \mid , \ \mid y'z'e'f' \mid \ \mid .$$

using 4×4 Laplace expansion.

The identity of compound determinants (a) and (b) was first given by Muir, who later * extended it.

When $n = 5$ the compounds turn out to be

$$| \ | a_1b_2 |, \ | c_1d_3 |, \ | e_1f_4 |, \ | g_1h_5 |, \ | x_2y_3 |, \ | x_2z_4 |, \ | x_2t_5 |,$$
$$| y_3z_4 |, \ | y_3t_5 |, \ | z_4t_5 | \qquad \dots\dots\dots\dots\dots\dots\dots(c)$$

and
$$| \ | xyzab |, \ | xytcd |, \ | xztef |, \ | yztgh | \ |, \qquad \dots\dots\dots\dots\dots(d)$$

where the non-compound consists of arrays $\left[\dfrac{a}{b}\right], \left[\dfrac{c}{d}\right], \left[\dfrac{e}{f}\right], \left[\dfrac{g}{h}\right]$, and x, y, z, t.

In arriving at (c) we consider the cofactor of a term such as

$$| a_1b_2 | \cdot | c_1d_3 | \cdot | e_2f_3 | \cdot | g_3h_4 |,$$

where no pair of suffixes is repeated.

This cofactor turns out to be a determinant which, after some subtraction of rows and columns, is a case of the non-compound for $n = 4$. We can therefore express it as a compound whose elements are of order 2 by using (a).

(d) depends on transposing the original determinant and expanding

$$\begin{vmatrix} x'y'z't' & & & & a'b' & \\ x'y'z't' & x'y'z't' & & & c'd' & \\ & x'y'z't' & x'y'z't' & & e'f' & \\ & & x'y'z't' & & g'h' \end{vmatrix}$$

by 5×5 Laplace.

It is seen that the transposed non-compound, of which the above is an example, is similar in form to the determinants used by Turnbull in deriving his fundamental identities.† More complicated identities ‡ exist similar to (a)-(b) and (c)-(d) involving higher order minors than the second, but the equivalent non-compound form is not evident. L. R. SHENTON.

2146. Sur le théorème de Feuerbach.

Dans un triangle ABC, soient A', B', C' les pieds des hauteurs AA', BB', CC' ; D, E, F les points de contact du cercle inscrit (I), de centre I, avec les côtés BC, CA, AB ; O, O_g les centres des cercles circonscrit (O) et des neuf points (O_g) ; H et Γ l'orthocentre et le point de Gergonne.

1. Les droites $B'C'$ et EF, $C'A'$ et FD, $A'B'$ et DE se coupent respectivement en M, N, P. Le triangle MNP est conjugué au cercle (I) et ses côtés sont tangents en A, B, C à l'hyperbole équilatère (H) qui passe par les points A, B, C, H, Γ § (Hyperbole de Feuerbach).

Or, le centre ϕ de (H), le centre O_g du cercle (O_g) et l'orthocentre I du triangle MNP sont collinéaires,‖ autrement dit les cercles (O_g) et (I) sont tangents entre eux au centre ϕ de l'hyperbole (H).

Le même raisonnement s'applique en remplaçant le cercle inscrit par chacun des centres des cercles exinscrits au triangle ABC, et le théorème de Feuerbach est démontré.

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* *Mess. of Maths.*, xlix.

† *Proc. Roy. Soc. Ed.*, Vol. XLIV, 1923–24, p. 55.

‡ *Determinants, Matrices, Invariants*, p. 47.

§ Ces remarques résultent de propriétés générales connues des coniques inscrites à un triangle.

‖ V. Thébault, *Mathesis*, t. LIV (Supplément, p. 38).

2. Le triangle MNP est conjugué au cercle (I) et à la conique (Σ) inscrite aux pieds A', B', C' des hauteurs, dont le centre est le point K de Lemoine du triangle ABC. La droite IK est le lieu des centres des coniques du faisceau (I), (Σ). La directrice de la parabole (P) inscrite au quadrilatère des tangentes au cercle (I) aux points D, E, F, ϕ est donc perpendiculaire à la droite IK et elle passe, en outre, par H et ω qui sont l'orthocentre du triangle ABC circonscrit à (P) et le centre du cercle circonscrit au triangle MNP conjugué à (P). Le centre ω du cercle circonscrit au triangle MNP est donc sur la perpendiculaire abaissée de l'orthocentre H du triangle ABC sur la droite qui joint le point K de Lemoine au centre I du cercle inscrit au même triangle.

3. Plus généralement, les céviennes d'un point arbitraire de l'hyperbole équilatère (H) inverse triangulaire d'un diamètre Δ du cercle circonscrit, rencontrent les côtés BC, CA, AB du triangle ABC en D, E, F. Si les côtés $B'C'$, $C'A'$, $A'B'$ du triangle orthique coupent les côtés EF, FD, DE du triangle DEF en M, N, P, le triangle MNP est circonscrit à l'hyperbole (H) en A, B, C et son cercle conjugué * touche le cercle des neuf points du triangle ABC en l'orthopôle du diamètre Δ.   V. Thébault.

2147. *Nombres premiers en progression arithmétique.*

Théorème. *Dans toute progression arithmétique ayant un nombre premier p de termes consécutifs, dont la raison n'est pas divisible par p, la somme des $(p-1)^{es}$ puissances des termes augmentée de l'unité est un multiple de p.*

En effet, soient
$$a, \; a+r, \; a+2r, \; \dots, \; a+(p-1)r$$

les p termes consécutifs de la progression arithmétique de raison r. Si l'on divise tous les termes par p, on obtient p restes inférieurs à p et tous différents entre eux qui sont donc, dans un certain ordre,
$$0, \; 1, \; 2, \; 3, \; \dots, \; p-1.$$
Il s'agit de démontrer que
$$S = 1^{p-1} + 2^{p-1} + 3^{p-1} + \dots + (p-1)^{p-1} + 1 = M \cdot p.$$

Or, d'après le théorème de Fermat, la puissance $(p-1)^e$ de tout nombre premier avec p, augmentée de l'unité, est multiple de p.

Les $p-1$ termes composant S sont donc égaux a l'unité plus un multiple de p, ce qui donne
$$S = M \cdot p + p - 1 + 1 = M \cdot p.$$

Corollaire. *p étant premier, si les p termes d'une progression arithmétique sont premiers absolus impairs, la somme de leurs $(p-1)^{es}$ puissances, augmentée de l'unité, est un multiple de p.*

Car, p est premier avec la raison r qui n'est divisible que par les nombres premiers non supérieurs à p et fait partie de la progression. †

Exemple.   7, 157, 307, 457, 607, 757, 907.
$$1^6 + 157^6 + 307^6 + 457^6 + 607^6 + 757^6 + 907^6 + 1 = M \cdot 7.$$
     V. Thébault.

2148. *Sur l'orthopôle.*

1. Théorème. *Dans un triangle ABC, le triangle $A_2B_2C_2$ ayant pour sommets les symétriques, par rapport aux milieux A_m, B_m, C_m des côtés BC, CA, AB, des points A_1, B_1, C_1 où les céviennes d'un point arbitraire P recoupent*

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* Ce cercle est toujours réel. (Ad. Mineur, *Mathesis*, 1937, 434.)

† V. Thébault, *Comptes-Rendus*, Paris. 1944. t. 218, pp. 223-224.

le cercle circonscrit et le triangle podaire X'Y'Z' du conjugué isogonal P' de P sont métaparallèles avec l'orthocentre H et l'orthopôle ϖ du diamètre OP du cercle circonscrit pour métapôles.*

Voici une autre manière de montrer que l'orthopôle ϖ du diamètre OP se confond avec l'un métapôles des triangles $A_2B_2C_2$ et $X'Y'Z'$ qui invoque un théorème peu répandu de G. Fontené.†

Soient G le barycentre du triangle ABC ; A_1', B_1', C_1' les points de rencontre des céviennes AP', BP', CP' avec le cercle circonscrit ; α, β, γ les milieux des cordes AA_1, BB_1, CC_1. On a, d'abord,

$$\frac{A_2A_m}{A_2A_1} \cdot \frac{\alpha A_1}{\alpha A} \cdot \frac{GA}{GA_m} = \tfrac{1}{2}. -1. -2 = +1.$$

Donc, en vertu de la réciproque du théorème de Menelaüs appliquée au triangle AA_mA_1 et aux points A_2, α, G situés sur ses côtés, ces trois points sont collinéaires et $GA_2 = -2G\alpha$, car G coïncide avec le barycentre du triangle AA_2A_1. De même, $GB_2 = -2G\beta$, $GC_2 = -2G\gamma$.

Le triangle $A_2B_2C_2$ est donc le transformé du triangle $\alpha\beta\gamma$ par l'homothétie $(G, -2)$.

De plus, la droite de Simson de l'orthopôle ϖ de OP, pour le triangle podaire $X'Y'Z'$ du conjugué isogonal P' de P, est parallèle à OP.‡ Il en résulte que les droites $X'\varpi$, $Y'\varpi$, $Z'\varpi$ passent respectivement par les centres O_x, O_y, O_z du cercles circonscrits aux triangles $X'yz$, $Y'zx$, $Z'xy$, x, y, z étant les points où le diamètre OP rencontre les côtés $Y'Z'$, $Z'X'$, $X'Y'$ du triangle $X'Y'Z'$. Les hauteurs $X'X''$, $Y'Y''$, $Z'Z''$ des triangles $X'yz$, $Y'zx$, $Z'xy$ sont donc les isogonales des droites $X'Ox$, $Y'Oy$, $Z'Oz$, c'est-à-dire de $X'\varpi$, $Y'\varpi$, $Z'\varpi$, dans les angles de ces triangles. Mais les médiatrices $O\alpha$, $O\beta$, $O\gamma$ des cordes AA_1, BB_1, CC_1 sont parallèles aux droites $Y'Z'$, $Z'X'$, $X'Y'$ qui sont donc perpendiculaires à $P\alpha$, $P\beta$, $P\gamma$. Les angles $(X'X'', X'Y')$ et $(PO, P\gamma)$ ont leurs côtés perpendiculaires et les angles $(PO, P\gamma)$ et $(\beta O, \beta\gamma)$ ont même mesure sur la circonférence décrite sur OP pour diamètre. On a donc les égalités d'angles

$$(X'Z', X'\varpi) = (X'X'', X'Y') = (PO, P\gamma) = (\beta O, \beta\gamma),$$

qui suffisent à établir le parallélisme des droites $\beta\gamma$ et $X'\varpi$, c'est-à-dire de B_2C_2 et $X'\varpi$, et, par analogie, celui de C_2A_2 et $Y'\varpi$, A_2B_2 et $Z'\varpi$. Le théorème est démontré.

Note. Dans le quadrilatère complet $X'Y'Z'xyz$, les droites $X'Ox$, $Y'Oy$, $Z'Oz$ concourent en l'orthopôle ϖ du diamètre OP. Dès lors, *le cercle de Miquel du quadrilatère complet X'Y'Z'xyz passe par l'orthopôle ϖ du diamètre OP du cercle circonscrit au triangle ABC.*

En particulier, *dans un triangle, le cercle de Miquel du quadrilatère complet déterminé par la droite des centres du cercle circonscrit et d'un des cercles tritangents, avec le triangle ayant pour sommets les points de contact de celui-ci sur les côtés, passe par le point de Feuerbach correspondant.*

2. THÉORÈME. *Dans un triangle, la somme des carrés des distances de deux droites rectangulaires à leurs orthopôles égale le carré de la distance de l'orthocentre au point de rencontre des droites de Simson parallèles aux deux droites.*

En effet, les orthopôles ϕ et ϕ' de deux droites rectangulaires Δ, Δ', dont la première coupe les côtés BC, CA, AB sous les angles de même sens α, β, γ, sont symétriques par rapport au milieu M de la distance PH du point de

* V. Thébault, *Comptes-Rendus*, Paris, 1944, 434–435.

† *Nouvelles Annales*, 1907, question 2077, et 1913, p. 137.

‡ V. Thébault, *Mathesis*, 1923, 144.

rencontre P de ces droites à l'orthocentre H du triangle ABC.[*] Leurs distances respectives à Δ et Δ' sont [†]

$$d = 2R \cos \alpha \cos \beta \cos \gamma, \quad d' = 2R \sin \alpha \sin \beta \sin \gamma,$$

R étant le rayon du cercle ABC. Donc,

$$PQ^2 = d^2 + d'^2 = 4R^2\{(\cos \alpha \cos \beta \cos \gamma)^2 + (\sin \alpha \sin \beta \sin \gamma)^2\}, \quad\ldots\ldots\ldots\ldots(i)$$

Q désignant le point de rencontre des parallèles à Δ, Δ' menées par ϕ, ϕ'.

Or, les droites de Simson parallèles à Δ et Δ' étant rectangulaires se coupent en Q' sur le cercle des neuf points du triangle ABC et passent respectivement en ϕ, ϕ', de sorte que le quadrangle $Q\phi Q'\phi'$ est un rectangle de centre M. Le point Q' coïncide avec le symétrique de Q par rapport au milieu commun de $\phi\phi'$ et PH ; donc $HQ' = PQ$, et le théorème est démontré.

COROLLAIRE. *Dans un triangle équilatéral, la somme des carrés des distances de deux droites rectangulaires à leurs orthopôles égale le carré du rayon du cercle inscrit.*

Car, l'orthocentre et les centres des cercles inscrit et des neuf points sont confondus ainsi que ces deux cercles.

N.B. Le relation (i) revient à dire que *si une droite Δ rencontre les côtés BC, CA, AB d'un triangle équilatéral ABC sous les angles de même sens α, β, γ, on a la relation*

$$(\sin \alpha \sin \beta \sin \gamma)^2 + (\cos \alpha \cos \beta \cos \gamma)^2 = \tfrac{1}{16},$$

qui se vérifie directement puisque $\beta = 60° + \alpha$, $\gamma = 120° + \alpha$, ou *vice versa*.

V. THÉBAULT.

2149. *A criterion for convergence of series and its applications.*

Let Σa_n be a series of positive terms, and suppose that by a selection or classification of all its terms we form from it m partial series. It is clear that the convergence of these m partial series assures the convergence of the given series, and conversely, and that this occurs, in particular, if m convergent series exist, each majorising one of the m partial series. But, in general, since the index attached to a term in the primitive series is different from the index it has as a term of that partial series to which it belongs, this process nearly always implies a comparison of a term in the given series with another which is not homologous to it in one of the majorants.

Let us now examine the case in which we have m convergent majorants such that the term a_n is less than or equal to its homologous term, the nth, in one of these series. The partial sum to n terms of Σa_n, for sufficiently large n, is less than the sum of the m partial sums (to the nth terms) of the majorants, and, since this itself is less than the sum of the values of the m majorants, the partial sums of Σa_n are bounded and the series Σa_n is convergent. Hence we have a test for convergence of series of positive terms, which, notwithstanding its simplicity, has unsuspected applications, of which we give two typical instances below.

The first consists in proving that *if the series of positive terms Σa_n converges, so does the series $\Sigma a^{1-\frac{1}{n}}$.* For divide the terms of the given series into two classes, including in the first class those for which $\sqrt[n]{a_n} \leqslant \frac{1}{2}$, and in the second the remaining terms. Since the condition $\sqrt[n]{a_n} \leqslant \frac{1}{2}$ implies $a_n^{1-\frac{1}{n}} \leqslant (\frac{1}{2})^{n-1}$, then to the first class there corresponds in the series $\Sigma a^{1-\frac{1}{n}}$ a partial series

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[*] V. Thébault, *Mathesis*, 1923, 144.

[†] J. Neuberg, *Bull. de l'Académie Royale de Belgique*, 1910.

with the geometric progression $\Sigma(\frac{1}{2})^{n-1}$ as majorant, and for the other part since $\sqrt[n]{a_n} > \frac{1}{2}$, we have

$$a_n^{1-\frac{1}{n}} = a_n / \sqrt[n]{a_n} < 2a_n ;$$

thus the other partial series, defined by the second class, is also convergent. The two majorants are then a geometric series of common ratio $\frac{1}{2}$ and the series $\Sigma 2a_n$.

For a second example, we prove that, for positive terms a_n, *if Σa_n converges, so does $\Sigma \dfrac{\sqrt{a_n}}{n^{\frac{1}{2}+\alpha}} (\alpha > 0)$.* For this, we group the terms of the latter series in two classes, according to whether they satisfy the first or second of the conditions

$$\sqrt{(n^{1+2\alpha}a_n)} > \tfrac{1}{2}, \quad \sqrt{(n^{1+2\alpha}a_n)} \leqslant \tfrac{1}{2}.$$

The partial series defined by the first class converges, since

$$\frac{\sqrt{a_n}}{n^{\frac{1}{2}+\alpha}} = \frac{a_n}{\sqrt{(n^{1+2\alpha}a_n)}} < 2a_n ;$$

and the second partial series also converges, since

$$\frac{\sqrt{a_n}}{n^{\frac{1}{2}+\alpha}} = \frac{\sqrt{(n^{1+2\alpha}a_n)}}{n^{1+2\alpha}} \leqslant \frac{1}{2n^{1+2\alpha}} .$$

The majorants now used are therefore the series $\Sigma 2a_n$ and the generalised harmonic series $\frac{1}{2}\Sigma(1/n^{1+2\alpha})$.

The reader will now be able to appreciate the extraordinary simplicity of these two illustrations of the method, and the advantages it gives over solutions obtained by employing different means.   V. INGLADA.

2150. *Some theorems on the parabola.*

1. The tangent drawn from the focus to the circumcircle of the triangle formed by any three normals is equal to the diameter of the circumcircle of the triangle formed by the three corresponding tangents.

This leads to an easy result when the normals concur.

2. If three normals be drawn to a parabola whose focus is S from a point P lying on the latus rectum, then SP is a diameter of the circumcircle of the triangle formed by the corresponding tangents.

3. If three tangents make angles θ_1, θ_2, θ_3 with the axis, then (a) if $\Sigma\theta = n\pi$ the circumcircles of the tangent triangle and of the normal triangle touch the axis and latus rectum respectively.

(b) if $\Sigma\theta = (2n+1)\pi/2$ the circumcircles of these two triangles touch the latus rectum and axis respectively.

4. If the circumcircle of the triangle formed by three tangents cuts the parabola in four points, the tangents at which make angles α_1, α_2, α_3, α_4 with the axis then $\Sigma\theta + \Sigma\alpha = (2n+1)\pi/2$, θ being as defined in 3.

5. If a circle drawn through one end of the latus rectum cuts the parabola again at P, Q, R, the triangle formed by the tangents at P, Q, R is congruent with the triangle formed by the normals at those points.

B. A. SWINDEN.

1641. What is it that kills a man in a collision? In his professional capacity Dr. Tattersall will know that it is the application of g (gravitation) more than the human frame can stand ; and he will also know that the amount of g depends on a combination of weight and speed on impact. It follows that road-users are potentially lethal in the order, motorists, motor-cyclists, cyclists, pedestrians ; and it behoves them to assume responsibility in that order.—Letter in *Daily Telegraph*, November 10, 1949. [Per Mr. W. G. Faires.]

REVIEWS.

Theory of Equations. By J. V. Uspensky. Pp. vii, 353. 27s. 1948. (McGraw-Hill)

It is generally realised that students of mathematics in American universities are not subjected to such a severe training in problem work as their opposite numbers in this country. But the above book, addressed primarily to American readers, is exceptional in this as well as in other respects for, among the many problems which it contains, those marked by asterisks are not only far from trivial, but frequently of considerable interest. The quality of these exercises can best be judged by one or two examples :

1. If the roots of the equation

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0$$

are real, then $a_i^2 \geqslant a_{i+1} a_{i-1}$, for $i = 1, 2, \ldots, n - 1$.

2. Find the smallest α such that the inequality

$$e^{-\alpha x} \leqslant \frac{1}{1 + x^2}$$

holds for all positive x.

3. By considering the equation

$$(x + a_1)(x + a_2)\ldots(x + a_n) = 0,$$

the a's being positive and not all equal, prove that

$$p_1 > p_2^{1/2} > p_3^{1/3} > \ldots > p_n^{1/n},$$

where p_i denotes the arithmetic mean of all products of i factors taken among $a_1, a_2, \ldots, a_n$.

This last example is worked out in the text and is noteworthy as generalising the classical Theorem of the Means due to Cauchy.

All the conventional topics are thoroughly covered in this volume and the treatment throughout is competent, rigorous and elegant. Very effective use is made of synthetic division, particularly in finding upper and lower bounds for the roots (is the word " bound " not preferable on all counts to " limit " in this connection ?), while the proof of Rolle's Theorem is made to depend on the logarithmic derivative of the polynomial in question.

Certain features which distinguish this work from others on the same subject deserve mention. For the separation of the real roots of an equation there is provided, in addition to Sturm's Theorem, an alternative in the shape of a remarkable result published by Vincent in 1836 * (seven years after the publication of Sturm's Theorem), hinted at earlier by Fourier and apparently subsequently overlooked. The method depends upon the simple idea that, if the roots of the equation $f(x) = 0$ are real and distinct, there is either one root or no root greater than a given integer λ, if λ be sufficiently large. The theorem asserts in general that, if the proposed equation be transformed by the chain of substitutions

$$x = a + 1/y, \quad y = b + 1/z, \quad z = c + 1/t, \text{ etc.}$$

where $a, b, c \ldots$ are arbitrary positive integers, then, after a certain stage, the coefficients in the transformed equation will present either one variation of sign or none. When combined with the operation of synthetic division the theorem enables us to construct a numerical scheme, called by the author a " genealogical tree ", by aid of which the separation of the roots can be readily effected.

* In the first volume of Liouville's *Journal.*

Matrices are freely employed and serve as a basis for the definition of determinants. Certain geometrical applications of the latter are given, including as a particular case the relation between the mutual distances of any four coplanar points in the form

$$\begin{vmatrix} 0 & l_{12}^2 & l_{13}^2 & l_{14}^2 & 1 \\ l_{12}^2 & 0 & l_{23}^2 & l_{24}^2 & 1 \\ l_{13}^2 & l_{23}^2 & 0 & l_{34}^2 & 1 \\ l_{14}^2 & l_{24}^2 & l_{34}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

While the book is elementary in purpose, wider issues are not entirely neglected. In the chapter on Symmetric Functions, for example, it is pointed out that " The algebraic solution of cubic and biquadratic equations as presented in Chapter V, although based on most elementary considerations, leaves the impression of a success achieved by the employment of ingenious artifices the reason of which is not clear. As in other cases, it is necessary to look at a problem from a higher standpoint in order to understand why an algebraic solution is possible for the cubic and biquadratic equations. And not only that : the same principles properly generalised and subjected to a more profound examination show the reason why an algebraic solution is generally impossible for equations of degree higher than 4 ". Nothing could be fairer than that! Furthermore, it is then shown in detail how it happens that, both for the cubic and biquadratic equations, it is possible to express the roots rationally in terms of a pair of radicals.

More difficult matters are relegated to five appendices. The first two deal with Gauss' fourth proof of the Fundamental Theorem of Algebra and Vincent's Theorem respectively. The third contains Hurwitz's proof that *the equation with real coefficients*

$$p_0 + p_1 x + p_2 x^2 + \dots + p_n x^n = 0$$

has all roots with negative real part if, and only if, p_0 and the n determinants

$$D_1 = p_1, \quad D_2 = \begin{vmatrix} p_1 & p_0 \\ p_3 & p_2 \end{vmatrix}, \quad D_3 = \begin{vmatrix} p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 \\ p_5 & p_4 & p_3 \end{vmatrix}, \dots$$

$$D_n = \begin{vmatrix} p_1 & p_0 & \dots & 0 \\ p_3 & p_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ p_{2n-1} & p_{2n-2} & \dots & p_n \end{vmatrix} \quad \text{are all positive.}$$

In the fourth appendix the calculus of matrices is applied to construct an iterative solution of the frequency equation for a dynamical system with n degrees of freedom—a problem which is equivalent to finding the latent roots of a symmetrical matrix. The last of these appendices is by no means the least valuable being devoted to Graeffe's Method for the solution of equations. This is also an iterative process depending upon successive squaring of the roots and does not seem to be as well-known as it ought to be. Its application is not confined to equations which are algebraic and it is a thoroughly practical procedure, particularly in a mathematical laboratory or whenever a multiplying machine is available. Indeed, it is probably the *only* practical method for computing the *complex* roots of an equation. The germ of the idea occurs in the writings of Waring in the eighteenth century, though it was adumbrated independently by Dandelin and Lobatchevsky. Its development in full detail was published by Graeffe in 1837.

A few trivial misprints are scattered throughout the text and there is something wrong with the first exercise on p. 117 ; but the production is excellent and the volume is well worthy of a place in every sixth form and university library.   T. A. B.

The Real Projective Plane. By H. S. M. COXETER. Pp. x, 196. $3. 1949. (McGraw-Hill)

This excellent book fills a gap in the ranks of geometrical textbooks. There are many on complex projective geometry, and some devote space to the axioms of real projective geometry, but few textbooks make the investigation of real projective geometry their main purpose.

Although one of Professor Coxeter's aims is to show that all the properties we expect to find in the real projective plane can be derived from suitable axioms, he is careful not to bore his readers at the outset by insisting too much on the purity of his intentions. His first chapter is therefore a light introduction. In the second, the axioms of incidence are introduced, the Principle of Duality established and harmonic conjugacy defined. With Chapter 3, on " Order and Continuity ", the fun begins. Six axioms of order bring us face to face with reality. We then have an axiom of continuity (replaced in Chapter 10 by a simpler axiom), and it is proved that " every opposite correspondence has exactly two invariant points ".

As an example, the following result is proved : " Let n given points have the property that the line joining any two of them passes through a third point of the set. Then the n points are collinear." This theorem is not true in complex projective geometry, as the configuration of the nine inflexions of a plane cubic curve indicates. It was enunciated by Sylvester in 1893, and remained unproved for about forty years.

Chapter 4, on " One-dimensional projectivities ", contains a proof of the fundamental theorem of projective geometry : " a projectivity is determined when three points of one range and the corresponding three points of the other are given." As the author says, this theorem " opens the way to the most characteristic developments of our subject ". From this point onwards we are therefore conscious of the general trend of the book some time in advance, but we are uniformly delighted, as in the earlier part, with the skill with which the author presents the subject matter.

Chapter 5, on " Two-dimensional projectivities ", is full of attractive results which it would be difficult to obtain algebraically without the elaborate apparatus of elementary divisors and canonical forms. As coordinates are not introduced until Chapter 11, a convincing case is made out for the teaching of real projective geometry before algebraic projective geometry. In Chapter 6 conics are introduced by means of polarities, after von Staudt. This method has the advantage of making the self-dual nature of conics self-evident.

Chapter 7 deals with projectivities on a conic and includes a neat proof of the theorem : " The joins of corresponding points of two projectively related ranges on a conic envelop a conic." The proof is ascribed to a former undergraduate at the University of Toronto, but may also be found on p. 284 of *Geometrical Conics* by F. S. Macaulay (1921 edition) and goes back to 1905. (It has always intrigued geometers that Baker, in Vol. II of *Principles of Geometry*, p. 52, prefers to prove this theorem by showing that the envelope is of class two.)

Chapter 8 deals with affine geometry, and Chapter 9 with euclidean geometry. Here again, there is much that is delightful, and all the familiar properties of circle and conic are obtained very easily. Chapter 10 is a short chapter on continuity. By a skilful use of the axioms on order, the terms " monotonic " and " limit " are defined, and the axiom of continuity of

Chapter 3 is replaced by " every monotonic sequence of points has a limit ". The real projective line is then shown to be perfect, and the fundamental theorem of projective geometry reconsidered in the light of the new set of axioms. Coordinates are swiftly introduced in Chapter 11, and new light is thrown on a number of concepts which are never explained very clearly in standard textbooks on analytical geometry—areal coordinates, for example. There is an appendix on the Complex Projective Plane, a Bibliography and an Index.

Enough has been said to show that this is an admirable book, and a delight to read. It has, in the reviewer's opinion, only one flaw : the author uses " wo " for " with respect to ", and says, in a footnote, " The preposition *wo* (pronounced like ' woe ') has been coined by some English mathematicians as a convenient abbreviation for ' with respect to ' or ' with regard to ' ". It is difficult to see why *wo* is more convenient as an abbreviation than " w.r.t. " ; if writing or saying " with respect to " makes too great a demand on time and energy, surely it would be better to adopt a symbol. It is interesting to note that in *An Introduction to Plane Geometry* Baker uses the symbolism C, D $\sim$ A, B to express the relation " C, D are harmonic conjugates with respect to A, B ", adding the warning that $\sim$ is not transitive. Perhaps Professor Coxeter will lighten our gloom by getting rid of *wo* from the new edition of his book which will surely soon be demanded. D. PEDOE.

Solid Analytical Geometry. By A. A. ALBERT. Pp. ix, 162. $3. 1949. (McGraw-Hill)

The successful writer of elementary textbooks is not usually an outstanding expert on the subject on which he writes. It is true that it is useful to know as much, or more, about the subject as is intended to be learnt from the book, but this is not an absolutely indispensable qualification. The real expert does not often succeed in writing a first-rate elementary text, and his books are, possibly because he sees further into the subject, usually much heavier than those of the professional textbook writer. For instance, Sir Horace Lamb's classical texts on Mechanics are rather tougher than those of his successors, of whom Sir James Jeans (whose text on the subject is too frequently overlooked) is about the only one who can compare with him in scientific stature. When the expert does score a resounding success it is mostly because he has something important to say about the direction which elementary teaching should take. It is this which accounts for the success of Hardy's *Pure Mathematics*, and, because he unburdened himself of what he needed to say in that book, he was under no compulsion to write another book of the same type. Professor Albert's *College Algebra*, although not so outstanding as Hardy's work, was also written for a real and urgent purpose and was the outcome of long and arduous cogitation on the questions with which it dealt. Unlike Hardy, Albert has not been able to resist the temptation of trying to repeat the success of that book and this new work is the result.

It would be unjust to Professor Albert to suggest that he is not attempting something useful in this book. His object is to bring about a simplification in the presentation of the subject by allowing himself a little more elaboration in the tools and techniques used. This is a highly commendable aim and is the basis of most advances in the presentation and digestion of mathematical theories, and of progress in most other forms of craftsmanship. Nevertheless, it does not appear to the reviewer that the attempt has clearly succeeded. The algebraic techniques used are not unfairly advanced and quite a lot of the exposition is worthy of careful study for its demonstration of the possibilities of the technique of transformation of coordinates. But the treatment is so heavy that the advantages are outweighed by the loss of that crispness and

geometrical feeling which should animate an elementary introduction to three-dimensional analytic geometry.

It is unfair to the author to give no examples of what the reviewer has in mind, and even more unfair to give all of them. Even a selection can justly be dismissed as unrepresentative, since the reviewer must naturally pick those which best fit in with his point of view. Perhaps it will be enough to draw attention to the chapter on "Spheres". This contains but two formal theorems. The first of these, whose presentation is perhaps lacking in punch, explains how to recognise the equation of a sphere. Theorem 2 states that "Two spheres intersect at the same angle at all points of the circle of intersection". The following excerpt from the proof of this theorem (the typography being exactly as in the original version) may cast some light on the reviewer's objection to the author's style.

$$" g = (a_1x_0 + b_1)(a_2x_0 + b_2) + (a_1y_0 + c_1)(a_2y_0 + c_2)$$
$$+ (a_1z_0 + d_1)(a_2z_0 + d_2) = a_1a_2(x_0{}^2 + y_0{}^2 + z_0{}^2)$$
$$+ (a_1b_2 + a_2b_1)x_0 + (a_1c_2 + a_2c_1)y_0 + (a_1d_2 + a_2d_1)z_0$$
$$+ b_1b_2 + c_1c_2 + d_1d_2 = \tfrac{1}{2}a_2[a_1(x_0{}^2 + y_0{}^2 + z_0{}^2)$$
$$+ 2(b_1x_0 + c_1y_0 + d_1z_0)] + \tfrac{1}{2}a_1[a_2(x_0{}^2 + y_0{}^2 + z_0{}^2)$$
$$+ 2(b_2x_0 + c_2y_0 + d_2z_0)] + b_1b_2 + c_1c_2 + d_1d_2."$$

D. B. S.

Geometry for Advanced Pupils. By E. A. MAXWELL. Pp. 176. 10s. 6d. 1949. (Oxford University Press)

This book should be a help to teachers who wish to stimulate the study of pure geometry by able pupils at the period, just after the stage of the old-style school certificate, when there is a temptation to make the course too exclusively analytical.

It is arranged in three sections each dealing with about 16 *configurations*. There are about 260 examples along with the configurations and 120 other examples taken from examination papers.

The treatment is original and should, as the author hopes, rescue the theorems from the isolation in which they often seem to stand.

In the first section, which deals chiefly with properties of the triangle, the threads of more elementary work are gathered together and some interesting new matter is added. Section II leads the reader gently from metrical ideas to a quasi-projective geometry resting on certain metrical props. The topics considered include duality, the line at infinity, ranges, pencils, projection, section, Menelaus and Ceva, Desargues, Pappus and involution. These lead on in section III to properties of the circle, many of which apply without modification to conics, and the configurations include those of the Pascal and Brianchon theorems, pole and polar, coaxal circles, and figures obtained by inversion ; but conics are regarded as just outside the scope of the book.

The author does not overlook the fact that there will usually be a concurrent course of analytical geometry, but believes that a short course of pure geometry has a value of its own in helping to trace a continuous argument from the simple beginnings into the richness afforded by projective geometry.

The reviewer finds the account of points at infinity quite unconvincing, and feels that this is because the subject (without the help of coordinates) is too difficult for a school class. While he welcomes the projective oases, he fears that the quasi-projective treatment may lead to misunderstandings that will be hard to remove at a later stage.

A. R.

Moderne Algebraische Geometrie. Die idealtheoretischen grundlagen. By W. Gröbner. Pp. xii, 208. $5.70. 1949. (Springer, Vienna)

Many who have been trained in the classical methods of algebraic geometry are at the present time looking anxiously for books which will help them to acquire competence in the modern algebraic methods which have recently become of the greatest importance in their subject. Within limits, they will find this volume a valuable aid, for it gives an introductory account of the ideal-theoretic treatment of varieties in a projective space defined over the complex field.

The book may be fairly described as a modernised version of Macaulay's Cambridge Mathematical Tract on " Modular Systems ". It begins with a straightforward account of the ideal theory of polynomial rings, and then goes on to the interpretation of the basic notions of ideal theory in terms of geometry. The dimension theory of varieties is then treated, by ideal-theoretic means, and after that postulation formulae and associated ideas are introduced by means of the Hilbert function, which is used most skilfully in order to obtain many of the standard results of the geometry of varieties in projective space. Thus, from the definition of the order of a variety from the leading coefficient of the Hilbert function, the author is able to prove Bézout's Theorem. Finally, he develops a syzygetic theory of homogeneous ideals by which he obtains a number of important geometrical results, including the fundamental theorem of Nöther.

The development of the theory of varieties by the methods used in this book is very ingenious and attractive, and a geometer will benefit by adding these methods to his equipment. Moreover, the exposition is good, and the process of learning will not be painful. But a most striking feature of the book is its complete neglect of the contributions of other geometers, such as Zariski and Weil. Some of the chief tools used by them, such as quotient rings and valuation theory, are not even mentioned, and even when the author's treatment leads him to give particular prominence to the notion of normal varieties —an idea of the highest importance to which Zariski has contributed so much —not a single reference is made to Zariski, even when he has proved the same or similar (and more powerful) results. The reader must therefore be warned that this volume is a somewhat one-sided account of modern algebraic methods in geometry, and that the study of it will not give him the full story of arithmetic methods in geometry. But with this caution it can be recommended as a useful addition to the literature of modern algebraic geometry.

W. V. D. HODGE.

Vorlesungen über höhere Mathematik. I. By A. DUSCHEK. Pp. x, 395. $7.80 ; geb. $8.70. 1949. (Springer, Vienna)

Dr. Duschek, who is Professor of Mathematics at the " Technische Hochschule " of Vienna, is perhaps best known in this country for his clear expositions of differential geometry and the tensor calculus, directed to the professional mathematician. His present work has a different aim : it is intended for physicists and engineers, but this does not mean that the author believes that " anything will do for the engineer ". On the contrary, he asserts that mathematics is mathematics, whether for the mathematician or the technician, and, further, that mathematics, if it is to be used with effect, cannot be accepted as a mere collection of rules and recipes ; it must be understood. The wide generality of the mathematician need not be sought, but logical proofs of theorems under reasonably restrictive hypotheses must be offered. This is no easy doctrine, but it is one which must be accepted by a larger proportion of our future technicians.

K

This first volume, of a promised set of four, is concerned chiefly with the calculus, for functions of one variable, generally real. The second volume is to contain the calculus for functions of more than one variable, and more algebra, on linear equations and determinants; the third will deal with ordinary and partial differential equations, with expansion in series and calculus of variations, the fourth with the fundamentals of function-theory. When the project is complete, it will be possible to institute a careful comparison with the joint (1944) report of the Institute of Physics and the Mathematical Association on the teaching of mathematics to physicists. This report, it will be remembered, decided to draw no distinction between pure and applied mathematics; but it would seem that Dr. Duschek's volumes will cover practically all the pure mathematics contained in the three schedules, A, B, C of this Report.

Much of the ground covered in the first volume is classical; we begin with the number fields, the integers, the rationals, the irrationals, the complex numbers; the elements of the theory of aggregates and the Bolzano-Weierstrass theorem on the existence of limit points; sequences and convergence. From here, the obvious path is to the doctrines of continuity and of limits, the calculus, its more elementary techniques and its immediate applications. All this is in fact dealt with, though only after a seeming digression about which something is said later in this review, and dealt with in a very lucid style, with some points of expository interest. For example, it is interesting to see that Dr. Duschek defines the logarithm as the integral of $1/x$, an order of development which Hardy introduced to teachers in this country some forty years ago, but which had not been very widely adopted in Continental textbooks. The derivative having been defined, the differential notation is explained, with clarity and indeed with emphasis, but without undue optimism, since the author himself remarks (p. 106): "In der Literatur findet man für die Differentiale dx und dy immer wieder die Benennung 'unendlich kleine Grössen'. Ich habe gegen diese Benennung schon einmal polemisiert. Aber sie erscheint unausrottbar ...". Evidently the controversy is not confined to the pages of the *Gazette*. Even more interesting is the fact that the first "calculus" chapter is headed "Das Integral und die Ableitung", and that the definite integral is first evolved from the concept of the area under a curve, that the definite integrals of x^2, of $\sin x$ and of x^n where n is rational, are evaluated, and that several of the basic properties of the definite integral are obtained, all before any mention of the derivative. This arrangement is not without good warrant, logical as well as historical, and its adoption in some recent books, as for instance in Ostrowski's new book on calculus, shows that there is a considerable body of well-informed opinion in favour of it. The traditional order often involves too great an emphasis on finding uncouth indefinite integrals, so that the pupil may have considerable trouble in learning that the definite integral is an important concept in its own right; the geometrical problems of drawing a tangent and of finding the area under a curve became prominent at about the same time, and the curious fact that one problem is the "inverse" of the other is striking and stimulating, but its effect is often lost by postponing the notion of the definite integral till late in the calculus course.

One unusual and prominent feature of this volume is the space given to probability, about fifty pages in all. First there is a section on the fundamental concepts, in which the author sets forth the classical theory, together with a summary of the criticisms which have been made of it; then we have the fundamental operations, effectively the addition and multiplication theorems. After the main calculus chapter, we get an interesting section on geometrical probability, then accounts of the mean and the standard devi-

ation of a weighted distribution, and finally the fundamental theorems of Bernoulli, Poisson and Bayes.

The final chapter of the volume groups together algebraic and calculus properties of polynomials and of rational functions, including numerical solution of equations, the elements of interpolation, and the integration of rational functions. Throughout the book there are scattered examples for the reader, not quite on the lavish scale of English textbooks, but enough for the purpose, and answers and hints are supplied at the end of the volume.

Altogether, this volume compels our admiration as a well-planned, finely balanced and clearly written account of the mathematics which a modern physicist, technician or engineer must know. We look forward with pleasure to seeing the remaining parts of the whole work, both for its own sake and for the purpose of comparison with the above-mentioned report. Finally, a word of thanks must be given to the printer, who has produced a really good piece of work in spite of post-war difficulties.   T. A. A. B.

Grundzüge der Theoretischen Logik. By D. HILBERT UND W. ACKERMANN. 3rd edition. Pp. 155, viii. DM. 16.50 : geb. DM 19.80. 1949. (Springer, Berlin)

Though it is generally regarded as the youngest of the mathematical disciplines, *symbolic logic* was conceived two and a half centuries ago by the same mind which fathered the differential and integral calculus, for it was Leibniz who first formulated the idea of a mathematical calculus by which the truth or falsehood of a proposition might be evaluated with the same facility as the answer to an addition sum.

Starting as a system of abbreviations, symbolic logic developed in the present century into a general theory of operations with signs. In the early eighteen hundreds Boole and de Morgan sought to condense what they held to be the *universal laws of thought* in an " algebra of logic ", as it was subsequently named ; today this viewpoint is entirely submerged, and the so-called laws of thought are regarded as forming an arbitrary assemblage of axioms having no greater *validity* (though perhaps much more *interest*) than any other set of axioms.

The Hilbert-Ackermann *Grundzüge der Theoretische Logik* was written in 1927 as an introduction to David Hilbert and Paul Bernays' projected two-volume work on the foundations of mathematics, but it soon proved to be of independent interest and value, and the second edition (1937) anticipated the appearance of volume 2 of the Hilbert-Bernays *Grundlagen der Mathematik* by nearly two years.

The four chapters of the *Grundzüge* take the reader from the elements of the propositional calculus to a fully fledged extended predicate calculus adequate for the expression and resolution of the paradoxes of the theory of classes. The first chapter describes the propositional calculus, introduces the fundamental notions of independence, completeness and freedom from contradiction for systems of axioms, and proves that the given set of axioms for the propositional calculus has all these properties. The great success of the axiomatic method in this field and the remarkable simplicity of the reasoning make this chapter an ideal introduction to axiomatics, well within the comprehension of the mathematical specialist in his first year at the University. The axiomatic foundation of the propositional calculus is the only one considered and there is no mention of the Wittgenstein truth-matrix method or of numerical evaluation methods (apart from their use in the proofs of independence).

The second and third chapters develop the first order predicate calculus which expresses the familiar notions of existence and universality by means of operators (Ex) and (x) applied to functions of an object variable x, the values

of the functions being propositions, and the problems of independence, completeness and non-contradiction are discussed in relation to the extended system of axioms.

The last chapter enters upon the much more difficult questions raised by the extended predicate calculus in which the universal and existential operators are free to act upon predicate variables as well as object variables. It is in this chapter that the principal change has been made in the third edition by the inclusion of a detailed formal theory of types. Another change is the omission of the brief bibliography which appeared in the second edition ; this omission is in itself of little importance but it is perhaps unfortunate that the reference to the full bibliography in the first volume of the *Journal of Symbolic Logic* has also been omitted.

The *Grundzüge* carries mathematical logic to the point where a cardinal is defined as a predicate of predicates and real numbers are introduced by Dedekind sections, but, faithful to its purpose as an introduction, stops short of the formalisation of arithmetic. The delight of discovering the astonishing part which multiplication plays in limiting the possibility of a proof of freedom from contradiction of a formal arithmetic is reserved for the reader whom the *Grundzüge* introduces to the *Grundlagen*.   R. L. GOODSTEIN.

La Géométrie et le Problème de L'Espace: IV. La Synthèse Dialectique. By FERDINAND GONSETH. Pp. 77. N.p. 1949. (Griffon, Neuchatel)

Nearly all mathematicians are more interested in teaching and creating mathematics than in talking about mathematics. The few who find themselves with a craving to talk about mathematics—what it is, what it is not, what it should be, . . . , are invariably transformed, after a very brief interval, into professors of education. They then spend most of their time talking about education—what it is, what it is not, what it should be. . . .

Very rarely, a mathematician in the full possession of his faculties writes about mathematics. The names of Klein, Poincaré, Polya, Hardy and Hadamard spring to one's mind. The result is usually stimulating and impresses the reader, anxious to read more, with the scarcity of writing of this kind.

Professor Gonseth's book is a welcome addition to this small collection of books on mathematics by mathematicians. Reading it, the professional geometer may note with surprise that the assumptions which most geometers take in their stride fill up quite an amount of space when set out in detail. But this detail is relieved by Professor Gonseth's easy style and abundant wit. In particular, his fable, " La boule dans la forêt," with its ingenious idea of a plan of the trees, which must be studied in an inn, conveniently situated on the edge of the forest, is very amusing.

We look forward to the forthcoming section on non-euclidean geometry, which will bring this adventure of geometrical ideas to an end.

                                                                        D. PEDOE.

The Structure of the Universe. By G. J. WHITROW. Pp. 171. 7s. 6d. 1949. (Hutchinson)

This book is the second to be published in the section of Hutchinson's University Library dealing with Mathematical and Physical Sciences. The publishers in their notice of the " Library " state : " The general aim of the whole series is to provide popular yet scholarly introductions for the benefit of the general reader, but more especially for the unprofessional student who wishes to pursue his chosen subject systematically up to something like a University standard. The books are written to be intelligible to those who have made no previous study of the respective subject ; . . . . The books

should be of special value to members of Adult Education Classes of all kinds, but it is believed that they will also be serviceable for undergraduates." It is obvious that to achieve this aim in the case of the subject dealt with by the present book is an extraordinarily formidable task. However, no one better qualified than Dr. Whitrow could have been found to make a success of it. Dr. Whitrow is intimately acquainted with the observational background of the subject. He is a recognised authority on its historical and philosophical aspects, and he has himself also made important contributions to the theories which he describes. In addition he is gifted with a very happy style of exposition for such a work ; without appearing to be didactic or unduly terse he contrives to convey an immense amount of information in as few words as possible, and does so in a way which maintains a freshness of interest throughout the whole book. Only those having some acquaintance with the literature of the subject can appreciate what a wealth of scholarship often lies behind a single sentence in Dr. Whitrow's presentation. The book is enriched by exceedingly apt quotations from a great array of other authors, including Pascal, Kant, Newton, Bishop Barnes, Edwin Hubble, Eddington and Milne, to mention only a few.

The first two chapters give a comprehensive historical account of the exploration of the universe with the telescope and spectroscope. They present a thoroughly up-to-date account of the observational facts concerning its structure. The next chapter is an historical account of the concepts of space and time, preparing the way for the following chapter on relativity. The latter gives a concise account of Einstein's special and general theories, their physical interpretations and observational tests.

The next two chapters are on " World-Models ". The first of them describes those resulting from general relativity, together with results of their comparison with observations. The second of them gives in the first place an account of Eddington's work, culminating in his posthumous work, *Fundamental Theory*. In a few pages the author conveys a clear impression of the spirit of Eddington's work and its attempt to relate the properties of the universe in the large to those of atomic systems. In the second place it summarises Milne's application of his kinematical relativity to produce a world-model, giving a lucid description of the ideas behind Milne's approach, and finishing with an explanation of Milne's introduction of two scales for measuring time.

The discussion of world-models naturally raises the question of the observational evidence regarding the age of the universe, since this evidence is expected to be an important factor in discriminating between claims of the various proposed models. Dr. Whitrow's next chapter gives a summary of the evidence provided by the interpretation of a great many types of astronomical observation. It gives a very clear statement of the existing situation in regard to these interpretations. The " structure of the Nebulae " (meaning by nebulae, the *galaxies*) is then dealt with. This chapter is an admirable description of the observational results on the distribution of the galaxies, and their dimensions, masses and structure, together with the theories of their dynamics due to Lindblad, Oort and Milne.

The final chapter is entitled " Cosmology and the *A Priori* ", and deals principally with the problems of natural philosophy raised by the approaches to cosmology followed by Eddington and Milne.

The author provides a useful bibliography, with brief comments on the items listed.

The treatment throughout, except for the quotation of a few simple formulae, is entirely non-mathematical. Many of the ideas have never before been expounded apart from their mathematical developments, and Dr. Whitrow has taken the reader very much further than one would have thought

possible under this limitation. Indeed, anyone proposing to try to follow mathematical work on any of the subjects dealt with would be very well advised to read first what Dr. Whitrow has to say about them, in order to get a grasp of the objective of the original work. Such a reader will have nothing to unlearn when he does come to master the more technical accounts. At any rate, this is true apart from one or two very minor arguments ; for example, on page 70 Whitrow states that " a natural clock on the sun . . . should run more slowly than the corresponding clock on the earth ". On the relativistic view which he is expounding this is true only if the two clocks are observed by a common observer, a point which he does not make quite clear. Again on page 133, where Whitrow mentions the determination of mass using Newton's theory of gravitation, the example he quotes gives the determination of the mass of the planet only in a rather indirect way.

A more general point, for which, however, Dr. Whitrow is not at all to blame, is that some of his discussions might have been somewhat modified had the ideas of Bondi and Gold, and of Hoyle, on the continuing creation of matter, and on the resulting possibilities of a steady-state theory of the expanding universe, been published before the manuscript of the present book was completed. It is too early to say anything very definite, but it does appear to the reviewer that the suggestions of these authors may resolve some of the difficulties of the subject of which Dr. Whitrow gives such a very fair statement in this book.

Altogether, the book can be recommended unreservedly to all the classes of reader for whom the publishers have designed the series. One can go a good deal further and say that many people who regard themselves as professionals in some parts of the subject will be exceedingly indebted to Dr. Whitrow for having produced such an accessible summing-up of the existing observational material and theoretical treatments.   W. H. McCREA.

Introduction to the theory of Fourier's series and integrals. By H. S. Carslaw. 3rd edition (rep.). Pp. xii, 368. $3.95. 1949. (Dover Publications, U.S.A.)

The Dover reprints have made available many classics which could not be re-issued in this country, Lamb, Love, Bateman, for example, and now a very old friend in Carslaw's book on Fourier series. Of the first edition, G. H. Bryan said, more than forty years ago : " Eminently suited to the mathematical student. As regards the student of physics, we would counsel him to have a copy of it in his possession." That the first part of this opinion can no longer stand unqualified is due to the enormous changes in the mathematical aspect of the theory flowing chiefly from Lebesgue's work : it is not merely that Lebesgue's theory enables us to give more satisfactory answers to certain questions, but that it has caused us to frame the questions themselves in a very different way. Professor Carslaw recognises this by his note on the third period of the history of the subject (1905 onwards) and by a thirty-page appendix on sets of points and the Lebesgue integral, which will serve to break the ice for the student who is contemplating a plunge into Hobson, Tonelli or Zygmund.

For the serious student of mathematical physics, anxious to have a firm grasp of Fourier theory as far as the Riemann integral will serve, Carslaw is still unsurpassed. The discipline is severe, for the author is uncompromising as an analyst. The proof of the main theorem, on Dirichlet's conditions, pivots on the second mean value theorem, while the concept of uniform convergence must be fully understood. Thus two hundred pages are devoted to the real number, limits, convergence and uniform convergence, the definite integral and its behaviour when the integrand depends on a parameter. The

remaining hundred pages apply these doctrines to Fourier series and integrals.

An ample supply of worked and unworked exercises and a liberal use of diagrams adds to the value of the text.   T. A. A. B.

Theoretical Hydrodynamics. By L. M. MILNE-THOMSON. Pp. xxviii, 600. £3. 1949. (Macmillan)

The first edition of this book, which appeared in 1938, has proved to be invaluable for teachers and students of theoretical hydrodynamics. This edition has been out of print for some years so that the second edition has been looked for with some impatience.

The general plan of the first edition (which was reviewed in the *Mathematical Gazette*, Vol. XXXIII, 1939, p. 102) is unaltered but some sections have been rearranged and re-written. The first five chapters which cover the general properties of fluid motion, and which give useful accounts of complex variable theory and vectors, are substantially the same, apart from minor changes such as the inclusion of a brief account of dyadics in the chapter on vectors. New features are included in the following chapters in which two-dimensional motion is studied systematically with the help of the complex variable. The disturbance of a given two-dimensional flow by the introduction of a circular cylinder is written down at sight with the help of the so-called " circle-theorem ", and, by conformal transformation, problems connected with cylinders of other cross-sections are solved just as easily. Improvements have also been made in the complex variable treatment of a moving cylinder in a fluid of infinite extent by using a conformal transformation $z = f(\zeta)$, which maps the region outside the cylinder in the z-plane on the region outside a unit circle in the ζ-plane. The inclusion of a source and a vortex in a compressible fluid makes an interesting addition to the two-dimensional work. A fallacious solution of the problem of the flow of a uniform stream past a circular cylinder in the presence of a wall has been omitted from Chapter 6 of the second edition.

Weiss's " sphere theorem ", the analogue of the " circle theorem ", is included in the chapters on the three-dimensional motions. Although this is an interesting general theorem it appears to be rather an elaborate method to use for solving the well-known problems of a sphere in a stream, or a sphere in the presence of a source.

In the chapter on viscosity dyadic notations are used for the stress tensor, and the idea of vector circulation over a surface leads to the derivation of lift and drag on a body in terms of the circulation and the inflow into the wake.

The most important change in the second edition is the inclusion of a chapter on the flow of compressible fluids at subsonic and supersonic speeds. This chapter contains some work which is a duplication of that given in the corresponding chapter on compressible flow in *Theoretical Aerodynamics* by the same author, and it is a pity that both accounts of compressible flow have not been combined in one book. In *Theoretical Aerodynamics* the emphasis is mainly on approximate methods, but the chapter in this book gives an introductory survey of exact results, including hodograph methods, characteristics, shock waves and intrinsic equations, and is a small, but welcome, addition to the book.

The considerable increase in price to £3 is regrettable, but probably unavoidable, and it is to be hoped that this will not prevent a wide use of an excellent book.   A. E. GREEN.

An Introduction to the Theory of Mechanics. By K. E. BULLEN. Pp. xvi, 368. 18s. 1949. (Science Press, Sydney)

This book covers the subjects of Dynamics, Statics, and Hydrostatics from their beginnings up to, roughly, H.S.C. standard. It does not include orbits,

catenaries, or attractions, but, apart from these sections, it would be of sufficient scope for the Applied Mathematics syllabus for the London B.Sc. External General degree. The absolute system with poundals and dynes is used throughout, but there are plenty of examples in which lb. wt. is used. The section on statics is considerably compressed : for instance, graphical statics including Bow's notation takes only $1\frac{1}{2}$ pages of bookwork and there are only four examples on light frameworks. No doubt it is right that there should be some reaction against the over-emphasis on statics which used to exist, but English teachers may feel that this is somewhat overdone in this book. The book has numerous sets of examples which appear to be excellent.

The special feature of the book, which distinguishes it from a number of others of similar scope, already on the market, is the continuous " running commentary " of hints and remarks which are interwoven with the text. At the end of every set of illustrative examples there is a set of remarks in smaller type, headed S.P.I.C.E., which the author translates as " Special points in choosing examples " [should not " choosing " here be " chosen "?]. These are just the kind of remarks that a good teacher will make to a class when going through a book of the usual kind. If you think yourself a good teacher you may think that they are superfluous, but even then you will probably get something out of them. If you are slightly doubtful of your capabilities, these remarks will be a godsend. If you are a student working on your own without a teacher, this is exactly what you want.

In order to make this more clear, it may be worth while to give a quotation. I select the remarks written at the end of three illustrative examples on impact and momentum. The first example is about two balls which collide and coalesce ; the second about shots fired from a recoiling gun ; the third about a number of railway trucks which are started up in turn as the couplings become taut.

" S.P.I.C.E. (α) Ex. (1) was chosen to show how to handle signs in using the principle of momentum. A diagram of the type of Fig. 23 is very useful in such cases.

(β) Ex. (2) was chosen in order to emphasise that, in using the principle of momentum, it is the ' absolute ' velocities that are involved in forming the moments of the particle of the system. If the data give a relative velocity, this must be connected with the ' absolute ' velocities as in equation (ii) of Ex. (2).

(γ) Ex. (3) was given as a problem involving several steps. The alternative method given in the note at the end is particularly instructive. The better student should watch for the possibilities of making a general step of this character.

(δ) The student might note that in part (iii) of Ex. (3) the mathematical analysis was facilitated by rationalising a denominator."

In addition to inserting these sets of remarks, the author has been at some pains to attempt to make the student look at mechanics as a part of a larger body of scientific learning. To this end, there are five chapters in addition to the main stream of bookwork and examples. Chapter I gives a brief discussion on Applied Mathematics in general, followed by a few " common-sense examples ". Chapters V, X, and XIV are labelled " Interlude " and give a résumé of work already done and work to come. Chapter XVIII, " a glimpse forward ", gives a short outline of further applied mathematics. I do not think that anything but good can come of these chapters. Most students will read them, I suppose, if only from curiosity, and they will help to give them some of the background that is so often lacking.

On the other hand, I think that Chapter XVII, " Appendix ", would have

been better omitted. This chapter gives a brief outline of vector products of vectors and of rotating axes, and I doubt the wisdom and usefulness of including this in a book of this scope. This chapter contains also a page of hints to examinees which are sound enough although somewhat trite.

A few minor criticisms may be made :

(1) Occasionally the author gives the impression of " showing off ". For instance, every chapter (and the whole book) starts with a quotation. Some are grave, some gay, and all seem out of place. Again, the author quotes Newton's laws in Latin and refuses to translate them. Whatever one's views as to the desirability of learning Latin, one does not want Latin thrown at one from a mechanics treatise.

(2) The printing is extraordinarily free from misprints, but is at times slightly irritating. The indices in the text seem a little too large and the italic letters in the figures have an exaggerated slope. Some of the blocks are really bad (e.g. fig. 53), and an odd footnote is one commencing with " I.e. " (sic).

(3) The author's insistence on the use of the word " absolute " seems to be a retrogressive step. In my opinion it would be better to replace this by " relative to the earth " in most of its occurrences.

(4) An " inclination of 1 in n " is generally taken to mean a slope whose sine is $1/n$. I am aware that there is a difference of opinion here, but the author on p. 102 definitely elects for the tangent.

(5) The symbol $\sim$ for " approximately equal " seems good, but is it standard? I would say that " gramme ", the author's choice, is less used than " gram ". E.S.P. instead of S.E.P. may be logically better, but again is unusual.

These criticisms are definitely minor ones. It would be impossible to write a book of this length and leave no loopholes for attack. On the whole, I think that this is a very good book indeed, and I thoroughly recommend it.

F. G. MAUNSELL.

A Shorter Intermediate Mechanics. By D. HUMPHREY and J. TOPPING. Pp. xii, 630. 15s 1949. (Longmans, Green & Co.)

Dr. Topping's use of the comparative " shorter " in the title immediately prompts the question, shorter than what with its 600 odd pages. And the answer is given in the preface ; the comparison is with Humphrey's two volumes of Intermediate Mechanics with about 1000 pages in all, on which Dr. Topping has based his revise. Some of the more advanced parts of Humphrey's, namely catenaries and virtual work, have been omitted and additional elementary examples included. The book was planned to cover the syllabus for the Intermediate Degree Examination in Applied Mathematics of the University of London, and now that that has been changed, it amply covers the Mechanics of the new syllabus and of the Advanced level Applied Mathematics of the General Certificate of Education.

Dr. Topping has obviously had prominently in mind the needs of the student preparing largely on his own without a great deal of help from class work, for there is a large number of worked examples—most important for this class of student—and these not only set out the mathematical steps, but include the sort of discussion of the problem and hints about the method of tackling it that would be given orally in a lecture. There is also a large number of examples with answers taken from up-to-date papers, and in a subject in which the fundamental principles are few, but their applications many and varied, these are essential for the student and together account for the 600 odd pages.

Any sequence of treatment has its difficulties and Dr. Topping has chosen

to start with the kinematics and then, through Newton's laws of motion, to deal with dynamics and to follow this with the statics and hydrostatics. But he advises that the order of reading be to leave the circular motion, simple harmonic motion and rigid dynamics until after the statics. This involves introducing ideas of centre of gravity and friction early, but saving full treatment until later. No doubt most experienced teachers have their own particular preference in order of treatment and the use of this book is not tied to the order given.

The two keystones in the building of a system of mechanics are (i) the treatment of force as a vector, and (ii) the linking of force with motion through Newton's laws. The treatment of the second of these is well done on the familiar and generally accepted lines with justifiable appeals to common sense and without the introduction of unnecessary units with queer names. A student trained on this book has no excuse for getting g on the wrong side of the equation. But in the treatment of force as a vector greater emphasis could have been given to the localised nature of the force vector in contradistinction to the non-localised couple vector. Section 204 would be found confusing to a student without some connecting explanation.

Some other treatments could have been improved by small additions ; the deeper significance of e, the coefficient of restitution, as the ratio of the impulse of recovery to the impulse of compression is not mentioned, and the more useful definition of the moment of a force as the arm times the resolved part perpendicular to the arm is not given prominence. In the work on stress diagrams the necessity of going round each vertex of the frame in the same way to get the loads in the right direction in the force diagram is not brought out, and in hydrostatics the connection between centres of pressure and centres of gravity of related shapes is omitted. No misprints have been detected, but fig. 54 is unfortunate in that the string should be longer, and the scale markings in fig. 185 are on a circle of the wrong radius. Both of these are reproductions from Humphrey's. There is no index, but as this will be used chiefly as a textbook rather than a reference book, this matters little ; and the topics treated in each chapter are set out fully in the contents table.

These are small blemishes in a book which will be attractive to the student and the teacher as it is well set out and deals with the subject in an interesting way. While essentially developing the theory logically, it does not disdain to mention everyday applications when they occur, thus enabling the student to subscribe to the dominie's description of mechanics as applied common sense. Further, although it has been written to serve preparation on the syllabus of an examination, the treatment is not narrowed by that purpose. In many places it goes beyond this syllabus thus providing an introduction and inspiration to the better student who will be going further.

E. F. BAXTER.

Mathematics.—A Textbook for Technical Students. By B. B. Low. 2nd Edition. Pp. vii, 464. 17s. 6d. 1949. (Longmans)

When the student and teacher usage of a textbook in mathematics is such that the second edition succeeds six separate impressions of the first in less than ten years it is almost unnecessary for a reviewer to commend its merits. This text for technical students may lack the logical sequence in the development of the subject beloved by the pure mathematician, but it has very great compensating merits in a directness of approach, a clearness of exposition, in readable print and clear diagrams, with the application of mathematics to sensible problems within the students' experience which is in the author's family tradition. We have heard the claim of " directly useful " used many times with less justification.

In this second edition the author rather modestly submits an extra chapter on Determinants to his wide circle of users. They will not be disappointed in his useful introduction to the subject—the chapter might well have been lengthened to include other methods of evaluating numerical determinants. All technical students seeking a reliable mathematical text should make the acquaintance of this book—they will find almost all their requirements satisfied within its covers ; non-technical students (if there be any such) would be well advised to add it to their library—they will have many opportunities of discovering its merits. W. Cooper.

Probability Theory for Statistical Methods. By F. N. David. Pp. ix, 230. 15s. 1949. (Cambridge University Press)

Dr. David is now at the Department of Statistics, University College, London, where she first studied under Professor Karl Pearson. But she worked also with Professor J. Neyman and it is to him that she dedicates this book. It is intended to state and prove those propositions and theorems of the calculus of probability that have been found useful for students of statistics. It is a good compilation and derives extensively from, in addition to the two statisticians just named, such others as Markoff, Uspensky and Cramer. This will perhaps indicate the nature of the field surveyed. The chapters deal with such matters as the binomial theorem, incomplete B-function, Poisson, the Lexis ratio, Markoff theorem, Tschebyscheff, and Liapounoff, and with the characteristic function and cumulants. There is also a somewhat isolated chapter on some simpler applications of genetical theories.

The book is therefore intended as a reference book. It is marred by rather more typographical errors than is common with this press, though all are very minor. The verbal text is not always very happy : thus the Dean of a Cambridge College who reports on the Number of Honours Grades is presumably not from Cambridge, England, whilst the phrasing is somewhat awkward in places, for example, on pp. 70, 180 and 205. The term " inversion theorem " of p. 196 is not, as far as we can see, defined or explained in the book. The bibliography is limited to listing at the end of each chapter a few of the principal authors and works : the reference (p. 222) to Cornish and Fisher is not sufficient as it stands for most readers.

But these are all relatively unimportant and can readily be corrected in another edition. Such will probably be found necessary, for there is no other similar collection of the kind available to give sound demonstrations of certain fundamental formulae in the field reviewed. Frank Sandon.

Sampling methods for Censuses and Surveys. By F. Yates. Pp. xiv, 318. 24s. 1949. (Griffin)

Much of statistical theory deals with sampling, but one of the origins of statistics—the census of population—has endeavoured to be " complete ", that is, to enumerate, etc., every individual of the universe involved, and not merely a sample thereof. Sampling has, however, always been used in such matters as crop reporting and its application in the recent developments such as Q.C. (quality control), market research, and public opinion surveys, is now well known. Less well known, perhaps, have been its applications to such important investigations as the National Farm Survey, carried out in England and Wales during the war, with the comparable Censuses of Woodlands and National Fruit Census, and the Wartime Social Survey (used, for example, by D.S.I.R. in its enquiry into Domestic Lighting and Heating). A great deal of work has been done on the technique and on the theory of such measurements and this book by Yates is an admirable review of the present position of this branch of statistics. Yates is particularly well qualified for

the work, for, as Head of the Statistical Department of Rothamsted Experimental Station, he was largely responsible for devising the methods of statistical analysis used in the National Farm Survey and was, in 1947, asked by the United Nations Sub-Commission on Statistical Sampling to prepare a manual in connection with the projected World Census of Agriculture and the World Census of Population. This book is apparently that manual.

The first five chapters (pp. 1–144) deal with the types of samples and the practical problems arising in connection with a survey, the last three (pp. 145–296) with the more mathematical part (Ch. 6, Estimation of the Population Values ; ch. 7, Estimation of the Sampling Error ; ch. 8, Efficiency) : the first formula in fact does not appear until p. 94. For all that the first part of the book is both original and interesting. Yates deals carefully with the natural unit (for example, in an Agricultural Survey is the unit the field or the farm?), the " frame " (a list or a map), with corresponding variations in procedure in random sampling, and with the question of bias. For all that, this is not quite easy reading for the non-technical reader, perhaps because Yates does not always give a definition but rather is inclined to assume that the reader knows what he means and goes on to illustrations. The various procedures for sampling—stratified random, multistage, interpenetrating, balanced, and so on—are all dealt with, and suggestions are made about their various advantages and disadvantages, many of which are the subject of more exact analysis in the three mathematical chapters. Thus it is of interest to note, from p. 129, this table from the National Farm Survey :

Size group (acres)	5–25	25–100	100–300	300–700	Over 700	Total
Average size (acres)	12	55	165	413	1035	—
No. of holdings	101450	111360	65210	11150	1430	290600
Sampling fraction per cent	5	10	25	50	100	—
No. of holdings in sample	5072	11136	16302	5575	1430	39516

" Had a uniform sampling fraction been used in place of a variable sampling fraction, a sample over twice as large would have been required to give results of the same accuracy " (on certain items). . . . " By the use of a variable sampling fraction results of ample accuracy were obtained from an analysis covering only one-seventh of all the holdings. . . . In consequence it was possible to make the results of the analysis available a year or two sooner . . ."

The aspect of the utility of the investigation is constantly before Yates, and an interesting development of this is to be found in the last chapter, the longest in the book. After a good review of " efficiency " (in the technical sense) with reference to analysis of variance and fiducial limits, Yates introduces a Cost Function to assist in a determination of the minimisation of the total cost of a survey and the corresponding optimal values of the sampling fractions : this is rather an intriguing linking-up with some of the ideas of mathematical economic theory.

The book concludes with two very short tables and with a long and useful bibliography (pp. 299–311) and an index (pp. 313–318). The bibliography is based on one prepared by the Statistical Office of the United Nations. The book is a valuable addition to the other texts of this publisher. Messrs. Griffin have now issued Yule and Kendall's *Introduction to the Theory of Statistics*, and Kendall's two-volumed *Advanced Theory of Statistics*, all reviewed in the *Gazette*, as well as Kendall's *Rank Correlation Methods* and the original Yule

(first edition, 1910). This is a record of a valuable service to statistical studies. Like the other books, this of Yates is well produced and misprints are rare. It can be recommended as a sound survey of its field, incorporating a great deal of widely scattered work, the greater part of which, it is of interest to note, originates either in Great Britain, in the U.S.A., or in India.

FRANK SANDON.

School Mathematics. A Unified Course. I. By H. E. PARR. Pp. xiv, 353. 7s. 6d. With Answers, 8s. 1949. (Bell)

This is a " Jeffery Report " book to cover the first two years from 11 + , of a course in three parts, to the standard of the present School Certificate. It is beautifully printed, and in price and bulk can compete favourably with the older method of separate volumes. There are many points for and against the unified course, purely from the point of view of cost, convenience and organisation. However, if the contents please, other considerations may not seem to matter much. The contents have an excellent feature in that the author has a fine sense of what are " do-able " examples. He has not cluttered up his book with examples nobody will ever use. It is even possible to set work from the book without previously vetting each example as regards suitability. These things cannot be said of some of our best known textbooks.

When the arrangement of the work is considered, both inside this one volume and with regard to what is chosen for this and what left for the other volumes, it seems fair to predict that every teacher will experience surprise, doubt, and, in fact, a variety of emotions. Is unity achieved by chopping up the study of decimals into five chapters, one of only a page and a half, and mixing them up with topics from the other subjects? The author's view, expressed in the preface, of an upper limit of " a week or so " per topic, seems reasonable if some emphasis is put on the " so ". His lower limit does not seem to fit in with the topic method at all. For lack of preliminary work Pythagoras' Theorem must wait till Part II for full treatment. Many will think the brief mention of it in Part I is a nuisance rather than a help.

The word " converse " may have been deliberately left out of this volume to be introduced later, but it does seem that the pupil is, for example to be left with the impression that point symmetry is always 2-fold. Of this latter type of blemish, the worst instance may be the very unsatisfactory treatment of significant zeros. H. B.

First Course in Arithmetic. By G. H. R. NEWTH. Pp. vi, 284. 4s. 6d. Without answers, 4s. 1949. (University Tutorial Press, London)

This volume is certainly " printed in large, clear type " and " provides a sound groundwork for pupils 11–13 ". The author selects this age-range to cover the " transfer " stage in the Secondary School, but we find questions dealing with " two trams passing each other " and " pipes filling a bath " which upset so many school certificate candidates. While one assumes that the author is thinking of the late transfer, it will be only the good boy who at 13, even from tables, can plot the parabola of the path of a body in flight through space, and read from the graph the maximum height, or again from tables plot an interest graph and read from it the interest after 17 years, or after what time the interest will have doubled itself.

The book is well produced and well planned. Each step forward follows in sequence and marks the author as a good practical teacher. He has worked to his belief that " practice on the part of the pupils themselves can bring accuracy and speed ", and we must agree when he says, " many make ' heavy going ' of their later work because they have not had sufficient practice earlier on."

The examples, in general, do not involve manipulation of large numbers, and there are quite sufficient to give plenty of practice. Many are topical and up-to-date. We have, " Princess Elizabeth was married on Thursday, November 20th, 1947. On what day of the week does Christmas Day fall in 1947? " " The population of China in 1944 was 482 millions. . . ." " The length of the Tay Bridge. . . ." " Basic petrol ration . . .". " The N.Z. four-masted barque Pamir." " Areas of Association and Rugby football pitches. . . ."

Explanations, diagrams and illustrations are very clear. There is what might be called an inconsistency in the matter of dealing with decimals, for in multiplication the " Standard form " is used, while in questions of division the dividend is adjusted for the divisor to be brought to a whole number.

While one agrees that the book will be found useful by Grammar School and Technical School pupils, it should prove excellent for all stages in the Modern Secondary Schools.   E. J. A.

Mathematics for Technical Students. Senior Course, second edition. By S. N. FORREST. Pp. viii, 323. 6s. 6d. 1949. (Arnold)

The first edition of this book was published in 1927 and was reprinted nine times, the last being in 1946. This is the first issue of the second edition which, like the original, aims to give the student, " such facility in Mathematics as will enable him to read technical books and journals with understanding . . . and attempt successfully such examinations as that for the National Certificate in Engineering ". The scope in this edition has been increased by one chapter dealing with logarithmic paper and Nomograms, and one on Complex numbers and Vectors, very valuable and useful additions.

A review of the Junior Course by the same author appeared in the *Gazette*, Vol. XXXIII (Feb., 1949). The Senior Course is a continuation of this. The earlier chapters cover Proportionals and Similar figures, Areas, Miscellaneous Equations, Indices, Surds and Irrational Equations, Logarithms, Variation and the Progressions, Circular Measure, Solution of Triangles and Trigonometrical ratios for the sum and difference of two angles. For the final stages a chapter is devoted to a " Further study of Graphs " and another, in 16 pages, covers Permutations and Combinations, the Binomial Theorem (including fractional and negative indices), Undetermined Coefficients and Exponential Functions, before concluding with the new chapters on Nomograms and Complex numbers.

At the end of the text and before the answers there are 12 pages devoted to tables of logarithms, etc.

The book in its first edition proved its value by its popularity, and this second edition should prove even more useful and valuable to those for whom it is intended.   E. J. A.

The Geometry of the Zeros of a Polynomial in a complex variable. By M. MARDEN. Pp. ix, 183. $5. 1949. Mathematical Surveys, 3. (American Mathematical Society)

This is an up-to-date account of a subject which, though very special in scope, has its own importance and charm, and has for this reason occupied the minds of many great and minor mathematicians from the times of Descartes until our present day. The main problems concern the position of the roots of a polynomial $P(z)$ in the complex plane or their number in a given part of it, and the position of the roots of the derivative $P'(z)$ (or some other related polynomial) relative to the position of the roots of $P(z)$; for example, the theorems of Lucas (1874) and of Grace (1902). This " analytical " branch of classical algebra is best characterised as an extension, into the complex

plane, of such familiar topics as the " sign rules ", the " Sturm sequences ", and the " theorem of Rolle ". The methods used are quite simple, the argument principle, especially in the form of Rouché's theorem, playing an important part.

The author, himself a leading expert in this field of research, has succeeded in giving an interesting and very readable survey of a subject whose literature is not easily accessible, in particular in the English language. A very exhaustive list of relevant references is another valuable feature.

W. W. ROGOSINSKI.

Analytical Geometry and Calculus. By FREDERIC H. MILLER. Pp. xii, 658. 30s. 1949. (John Wiley, New York ; Chapman & Hall)

In this excellent book by Professor Miller of the Cooper Union School of Engineering, " the subjects of plane and solid analytic geometry, differential calculus, and integral calculus have been correlated as a single branch of mathematical analysis ", an obvious convenience for the scientist and engineer, for whom the book is primarily intended.

Chapter 1 explains what is meant by a function and deals briefly with graphs and tests for symmetry. In Chapter 2 the usual straight-line formulae are obtained. Chapters 3 and 4 introduce the limit concept and rules for differentiation. Teachers will find Chapter 5 particularly interesting, for in it, integration is introduced " by summation " before it is otherwise mentioned, and after the law of the mean for integrals has been obtained, the fundamental theorem $\dfrac{d}{dx}\displaystyle\int_a^x f(z)\,dz = f(x)$ is deduced from it. That integration by summation must be introduced into any calculus course at the earliest possible moment is unquestionable, but as a first approach I think it might well prove too difficult for many students. In Chapter 6, calculus is applied to problems involving maxima and minima, rates of change, areas and volumes.

Chapters 7, 8 and 9 deal with circles, transformations of coordinates and conics ; the applied mathematician should find the treatment adequate.

Chapter 10 introduces inverse and exponential functions, and following a chapter on methods of integration, further chapters deal with parametric equations, indeterminate forms, centroids, moments of inertia, etc. Chapter 15 gives a good account of polar coordinates and their applications.

Chapters 16 and 17 are devoted to three dimensional analytical geometry and the treatment of planes, quadrics and curves in space is full and clear. The introduction of partial differentiation in Chapter 18 enables the author to obtain the equations of tangent planes to surfaces and to investigate singular points of plane curves.

Chapter 19 deals with double and triple integrals, and Chapter 20 with the convergency of infinite series, followed by a proof of Taylor's Theorem. Various tables of useful formulae and " standard forms " round off the book.

There are many worked examples in the text and 3025 are provided for the student to work, answers to the odd-numbered ones being supplied.

Throughout the book calculus and analytical geometry are nicely interwoven, though the " correlation " claimed is likely to be more significant to the American than to the English student. For analytical geometry books written for the former frequently do without calculus altogether, whereas in this country we should expect to find it used ; and further, the many applications to geometrical problems which the author gives would be found in most calculus books of comparable standard originating here.

Although the text is written for the applied mathematician, which justifies

the exclusion of large and important chunks of geometry (pole and polar, for example), points of importance to the pure mathematician are not glossed over ; when assumptions have to be made, they are clearly stated.

It is perhaps surprising to find no mention of differential equations, and, even in the exercises, no emphasis on the applications of the text to engineering and physical problems. The explanation may be that these are to be found in *Advanced Mathematics for Engineers*, of which Professor Miller is co-author.

The book is beautifully printed and bound.   R. W.

Geometrical Tools. A Mathematical Sketch and Model Book. By R. C. YATES. Pp. 194. \$3.50. Revised edition. 1949. (Educational Publishers, St. Louis, Missouri)

The first (1947) edition was warmly welcomed in the *Gazette* (XXXI, p. 186) and school libraries were advised to try to get copies. This advice stands. The new edition is much the same as the first in content, but the production, apparently by photography from varityper script, is much neater and more pleasing than in the original edition ; the diagrams too are clearer.

The constructional side of geometry, with the straightedge, the compass, paper folding and linkages as the practical tools, is set out in a most stimulating fashion, from the basic constructions of elementary geometry up to Kempe's theorem, that a linkage can be devised to construct any algebraic curve. The volume, so the author tells us, was designed to help intending teachers of mathematics ; it should certainly whet the appetite of the novice, and a teacher would have to be extraordinarily unlucky if he failed to extract numerous valuable hints and ideas.

The author, Colonel Yates of the U.S. Military Academy, has achieved a happy fusion of theoretical with practical geometry, each constituent in its due proportion ; this wholesome balance is perhaps partly due to Colonel Yates' experience as a Service instructor, but it is a characteristic which must be an essential part in the planning and teaching of our courses in technical colleges, junior and senior.

In spite of devaluation, teachers should make every effort to get this book for their own reading and for the shelves of the school library.

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